More adventures in Binary Land

October 17, 2009

New notation: The following numbers are called *powers of* 2*:*

$$2^1 = 2, \quad 2^2 = 2 \cdot 2, \quad 2^3 = 2 \cdot 2 \cdot 2, \quad 2^4 = \underbrace{2 \cdot 2 \cdot 2 \cdot 2}_{4 \text{ times}}, \qquad \dots .$$

In general, nth power of 2 is obtained by multiplying 2 by itslef n times. Let's also agree that $2^0 = 1$.

Digits of numbers in binary (base 2) notation have a square around them. Digits of numbers in the usual decimal (base 10) notation do not have squares around them.

Decimal (base 10) notation and Binary (base 2) notation

Decimal notation for numbers is the usual notation. The word *decimal* refers to the base of the system, which is 10.

The number 2 plays the same role in Binary Land as the number 10 plays in the usual decimal system.

$$\begin{array}{lcr lcr} 2^1 = 2 & = & \boxed{1}\boxed{0} & 10^1 & = & 10 \\ 2^2 = 4 & = & \boxed{1}\boxed{0}\boxed{0} & 10^2 & = & 100 \\ 2^3 = 8 & = & \boxed{1}\boxed{0}\boxed{0}\boxed{0} & 10^3 & = & 1000 \end{array}$$

Example in decimal notation:

$$\begin{aligned} 503 &= \mathbf{5} \times 100 + \mathbf{0} \times 10 + \mathbf{3} \times 1 = \\ &= \mathbf{5} \times 10^2 + \mathbf{0} \times 10^1 + \mathbf{3} \times 10^0. \end{aligned}$$

The value of each digit is indicated below:

$$\begin{array}{ccc} 5 & 0 & 3 \\ \uparrow & \uparrow & \uparrow \\ 10^2 & 10^1 & 10^0 \end{array}$$

Note that the digit on the right has value equal to itself (i.s., corresponds to $10^0 = 1$).

The same is true in binary notation. For example, for the number $\boxed{1}\boxed{1}\boxed{0}\boxed{1}$ in binary notation we have

$$\begin{array}{cccc} 1 & 1 & 0 & 1 \\ \uparrow & \uparrow & \uparrow & \uparrow \\ 2^3 & 2^2 & 2^1 & 2^0 \end{array}$$

This means that

$$\boxed{1}\boxed{1}\boxed{0}\boxed{1} = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 8 + 4 + 1 = 13.$$

1. Convert binary notation into decimal using the place value of the digits (as above):

   - $\boxed{1}\boxed{1} =$

   - $\boxed{1}\boxed{0}\boxed{1}\boxed{1} =$

- $\boxed{1}\boxed{0}\boxed{0}\boxed{1}\boxed{1}$

2. How do powers of 2 look like in binary (base 2) notation?

3. In decimal notation, we are counting by *groups of* 10s, *groups of* 100s, and so on.
   For example, 523 means that we have 5 groups of 100, 2 groups of 10 and 3 ones.
   Since we can have up to 9 ones before we go to a group of 10, we need 9 digits in the decimal notation.
   Explain why we need exactly 2 digits in the system with base 2.

4. Let's see how subtraction is done in binary notation:

$$
\begin{array}{rcccc}
- & & \boxed{1} & \boxed{1} & \boxed{1} \\
 & & \boxed{1} & \boxed{0} & \boxed{1} \\
\hline
 & \square & \square & \square & \square
\end{array}
$$

Check this in decimal notation:

$$\begin{array}{r@{\quad}c@{\quad}c@{\quad}c@{\quad}c} - & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{0} \\ & & \boxed{1} & \boxed{0} & \boxed{1} \\ \hline & \square & \square & \square & \square \end{array}$$

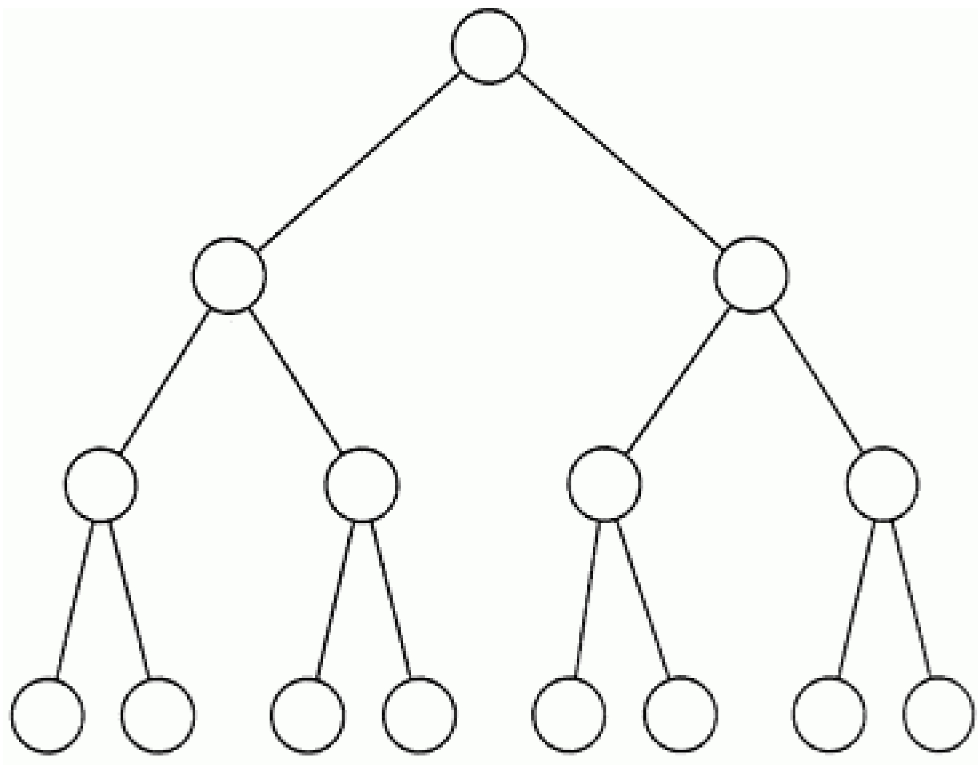
Check this in decimal notation:

$$\begin{array}{r@{\quad}c@{\quad}c@{\quad}c@{\quad}c} - & & \boxed{1} & \boxed{0} & \boxed{0} \\ & & \square & \boxed{1} & \boxed{1} \\ \hline & \square & \square & \square & \square \end{array}$$

Check this in decimal notation:

$$\begin{array}{r@{\quad}c@{\quad}c@{\quad}c@{\quad}c} - & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{0} \\ & & \square & \boxed{1} & \boxed{1} \\ \hline & \square & \square & \square & \square \end{array}$$

Binary Tree



1. Label the vertices of bottom row by numbers 0 through 7 (going from left to right). (You may put numbers right inside of the circles).

2. Play the game "Guess my number around" (with numbers from 0 to 7). The goal is to be able to guess the number from 3 attempts (or less, if you are lucky). Can you think of how the edges can help you to formulate the strategy? What is the best way to play the game?

3. Label all the edges pointing to the left by 0;
   Label all the edges pointing to the right by 1.

4. For each number on the bottom, there is exactly one path from the top circle down to this number. Mark the path from the top circle to the number 3. Mark with a different color the path from the top circle to the number 6.

5. For each number, the path connecting the top circle with this number gives you a string of 0's and 1's. (Start writing down the path from top to bottom)

   (a) Write down the string of 0's and 1's along the path going to 2:

   (b) Write down the string of 0's and 1's along the path to 6:

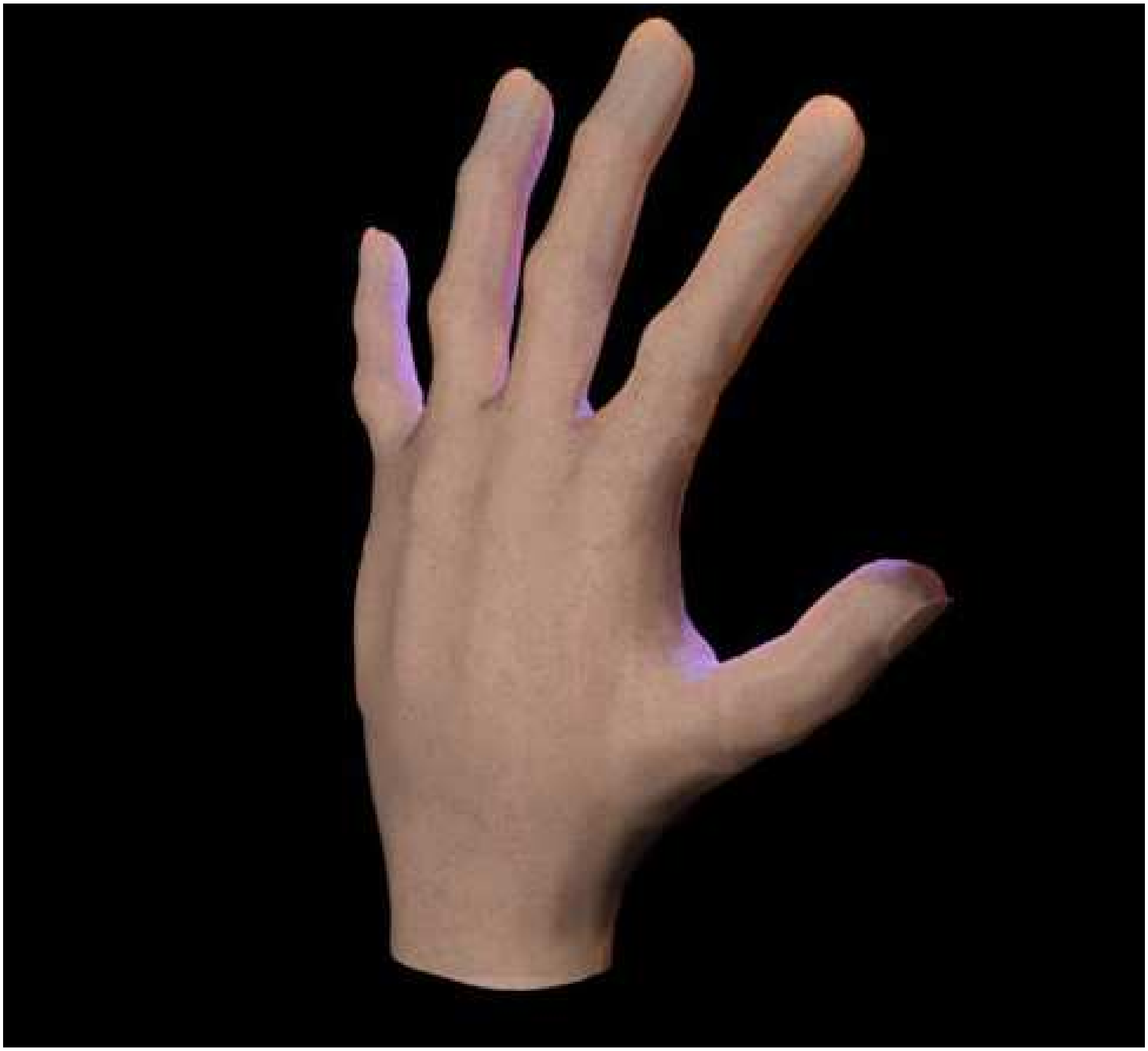
   (c) Do you recognize your answers in (a) and (b)?

6. How many questions do you have to ask to guess the number in the "Guess my number" game with numbers ranging

   (a) from 0 through 15?

   (b) from 0 through 31?

How high can you count on fingers of one hand?



Use your left hand fingers to count up to 31:

- Each finger represents a binary digit of a number;
- If a finger is up, the digit is 1; if a finger is down, the digit is 0.

Put your left hand on the table (palm down). Label the fingure on your left hand as follows:

- thumb = 1
- index finger = 2
- middle finger = 4
- ring finger = 8
- pinky = 16

For each of the binary numbers, implement them on your left hand (press the hand on the table as you are doing this to make it easier). Then add together the values of all the digits to get the decimal number:

1. | 1 | 1 |

2. | 1 | 0 | 0 | 0 | 1 |

3. | 1 | 1 | 0 | 1 | 1 |

4. Since you have 5 fingers on one hand, you can represent binary numbers that require no more than 5 digits. What is the biggest number you can represent this way?

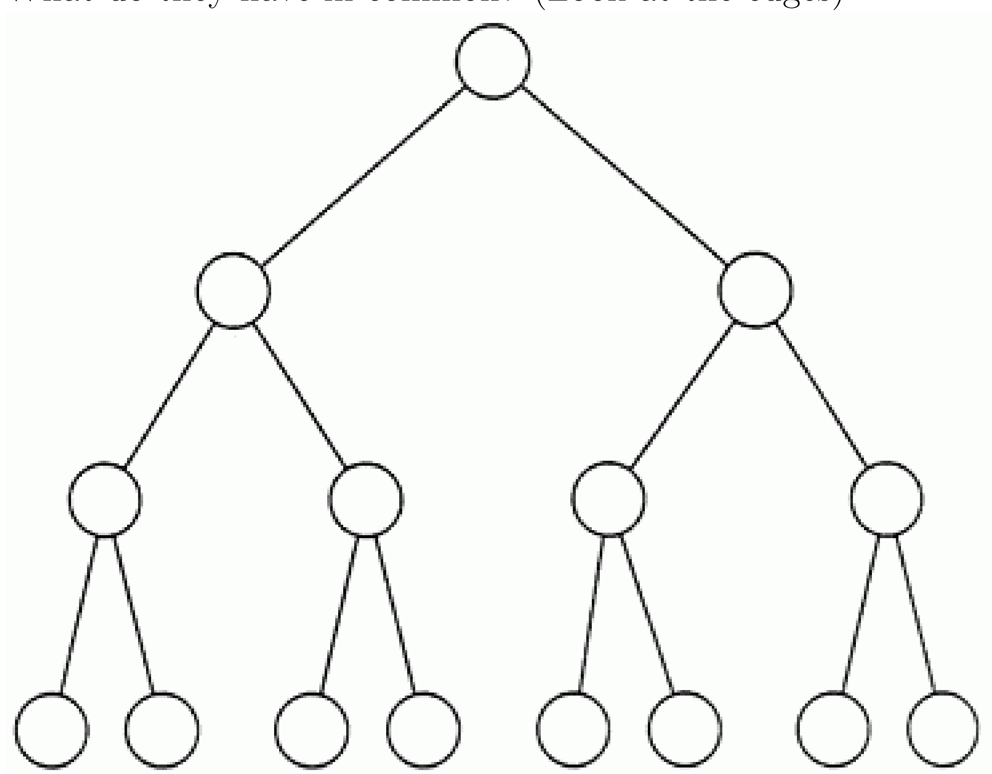
A card trick

I have 4 cards with numbers 1 through 15 written on them. (The same number appears on several cards). Here are the cards:

$$\begin{pmatrix} 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{pmatrix} \begin{pmatrix} 4 & 5 & 6 & 7 \\ 12 & 13 & 14 & 15 \end{pmatrix} \begin{pmatrix} 2 & 3 & 6 & 7 \\ 10 & 11 & 14 & 15 \end{pmatrix} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 9 & 11 & 13 & 15 \end{pmatrix}$$

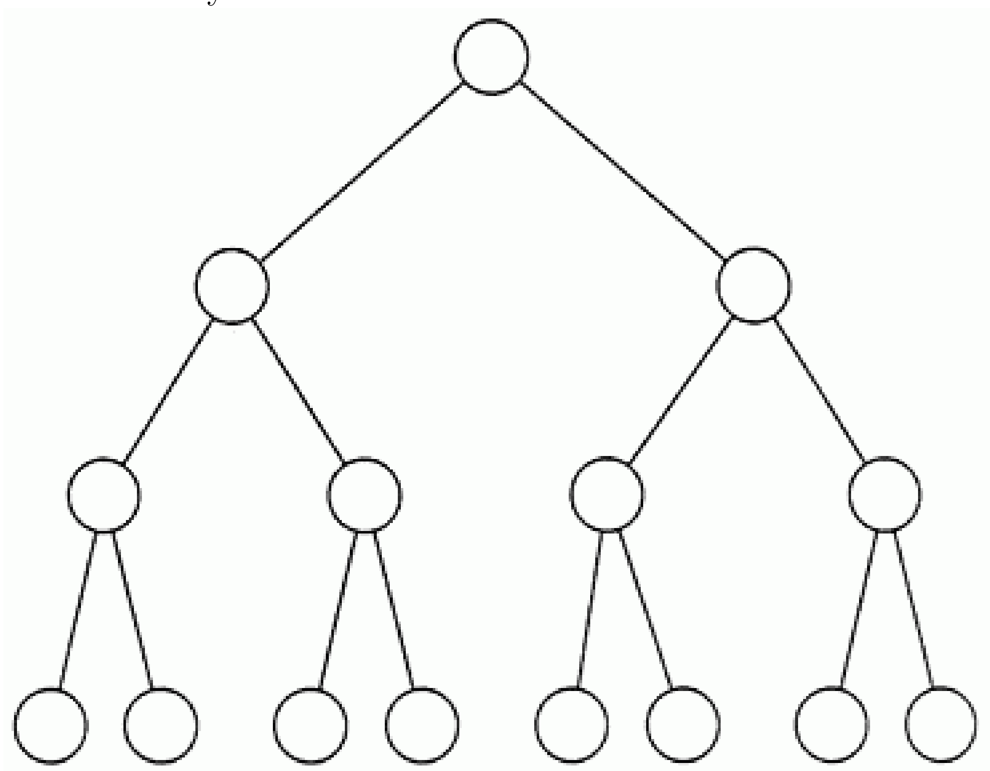
All these numbers can be writte in binary notation using 4 digits (for some numbers, the first digit(s) can be 0s).

1. On the binary tree below mark the numbers appearing on Card 1. What do they have in common? (Look at the edges)



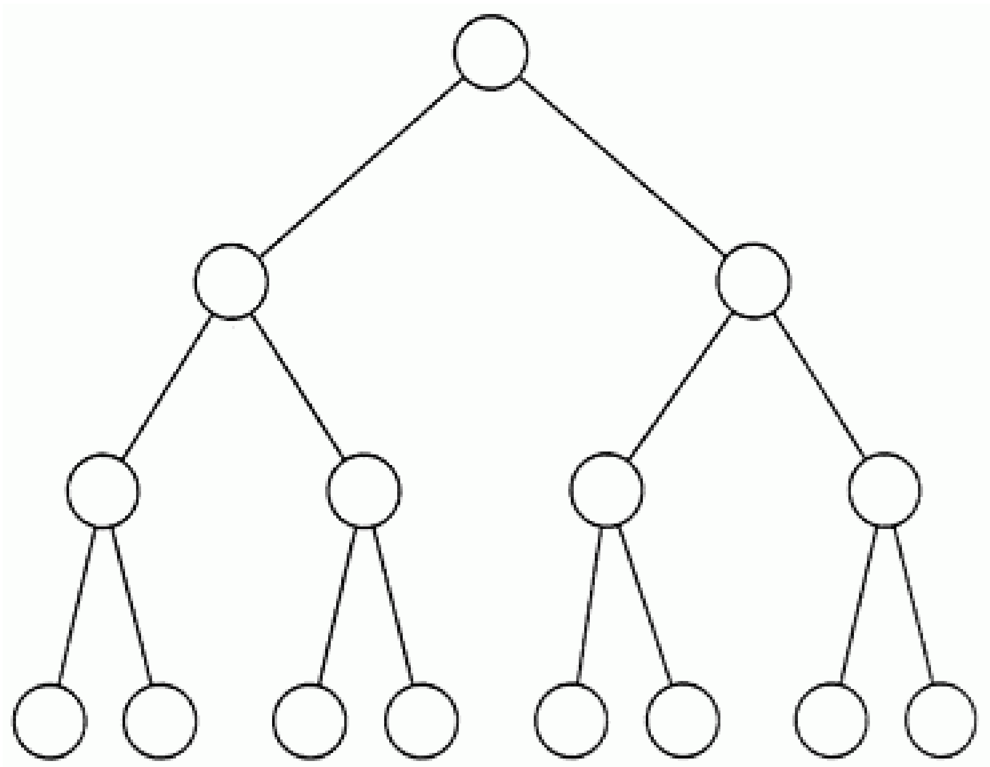
2. What is the first digit of all the numbers appearing on Card 1?

3. On the binary tree below mark the numbers appearing on Card 2. What do they have in common?

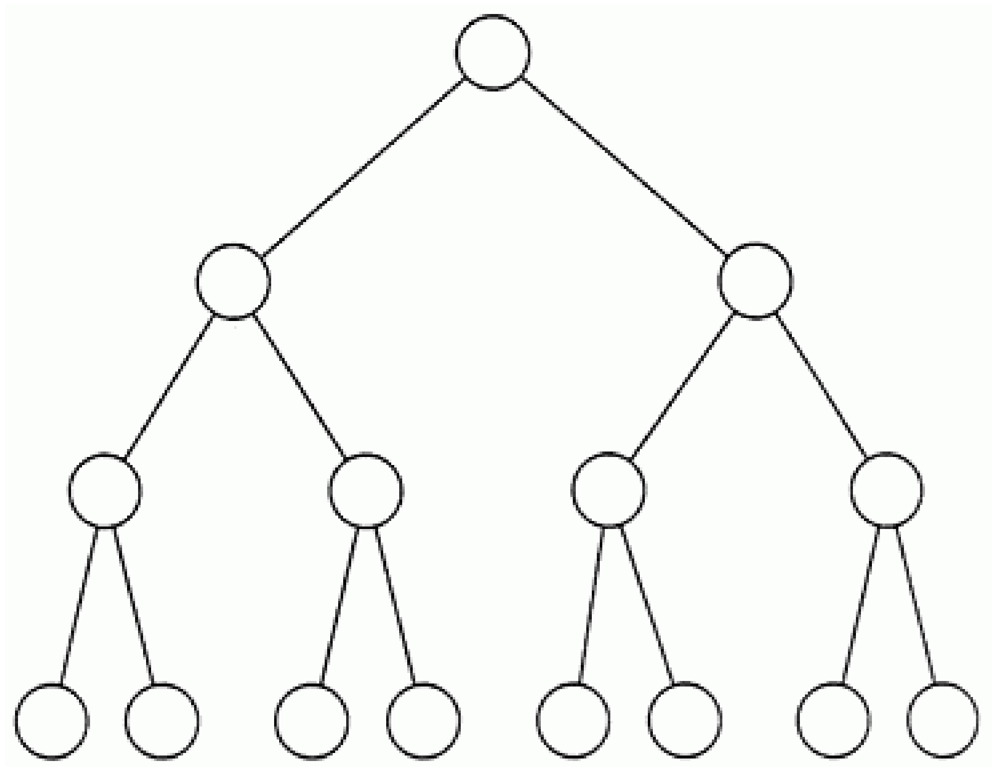


4. What is the second digit of all the numbers appearing on Card 2?

5. On the binary tree below mark the numbers appearing on Card 3. What do they have in common?



6. What is the third digit of all numbers appearing on Card 3?
7. On the binary tree below mark the numbers appearing on Card 4. What do they have in common?



8. What is the last digit of all numbers appearing on Card 4?

9. How can you tell what number you have given the numbers of the cards it is written on?