

Composite Numbers, Primes Numbers, and 1

The **natural numbers** are the numbers $1, 2, 3, 4, 5, \dots$, and so on. Among natural numbers, we distinguish three types of numbers:

1. **Composite numbers:** These are numbers that can be written as a product of two smaller numbers. For example, $6 = 2 \cdot 3$.
2. **Prime numbers:** Everything that isn't composite, except 1, is a prime number. Thus a prime number can not be written as a product of smaller numbers. For example, 5 is a prime number.
3. **Numbers that are 1:** There is only one such number. The number 1 is neither prime nor composite.

Starting with a composite number, we know (since it's composite) that we can break it down into smaller pieces. For example, if we take the number 520, we can break it down into the factors 52 and 10. Each of these decomposes further, for example $52 = 4 \cdot 13$ and $10 = 2 \cdot 5$. We can break down 6 a little more, into $6 = 2 \cdot 3$, and then rearrange everything that's left into increasing order:

$$520 = 52 \cdot 10 = 4 \cdot 13 \cdot 2 \cdot 5 = 2 \cdot 2 \cdot 13 \cdot 2 \cdot 5 = 2 \cdot 2 \cdot 2 \cdot 5 \cdot 13.$$

It's clear that we can keep going until we can't go any more. The numbers we are left with are numbers that can't be broken into smaller factors, and we call these **prime numbers**.

What if we start differently? We could start with $520 = 8 \cdot 65$, and keep going to get

$$520 = 8 \cdot 65 = 2 \cdot 4 \cdot 5 \cdot 13 = 2 \cdot 2 \cdot 2 \cdot 5 \cdot 13.$$

Of course, if we start with a prime number instead of a composite number, there's nothing to do.

What **is not** clear is that given any number, this process of decomposition, no matter how it is performed, results in a unique answer. This fact deserves a special name:

The Fundamental Theorem of Arithmetic: Any natural number different from 1 can be uniquely represented as a product of primes in increasing order.

Some Divisibility Problems

If a and b are positive integers, then we say that b **divides** a and also that a **is divisible by** b , if there is a number q such that $a = qb$. The number q is the **quotient** of a by b . To write down “ b divides a ” with symbols, we write $b \mid a$. When b does not divide a , we write $b \nmid a$.

Examples: $4 \mid 12$, and $13 \mid 91$, but $12 \nmid 25$.

- Which of the following numbers divides $2^9 \cdot 3$:
(a) 2 (b) 5 (c) 8 (d) 9
Why? (Or why not?)
- Which of the following is true (and why!):
(a) If a number is divisible by both 3 and 4, then it must be divisible by $3 \cdot 4 = 12$.
(b) If a number is divisible by both 4 and 6, then it must be divisible by $4 \cdot 6 = 24$.
- (a) If a number N is not divisible by 3, is it possible that $2N$ is divisible by 3?
(b) The number $15N$ is divisible by 6. Does N have to be divisible by 6?
- (a) Prove that the product of three consecutive natural numbers is divisible by 6.
(b) Prove that the product of five consecutive natural numbers is:
 - ... divisible by 30.
 - ... divisible 120.
- Check whether the number 24681357975318642 is divisible by:
(a) 2 (b) 4 (c) 5 (d) 10 (e) 3 (f) 9 (g) 11 (h) 7
What about a number which consists of a googol 1's followed by a googol 2's? (A googol is a 1, followed by one hundred 0's.)
- An agent at an incompetent intelligence agency was using a code where each number is assigned a distinct letter of the alphabet. The agent writes $AB \times CD = EFFF$. Prove that the agent is wrong.

Two Related Problems

1. A number of pins are evenly spaced around a circle. A thread is tied to one pin, and looped tightly around a second pin. The thread is then looped tightly around a third pin so that the clockwise gap between the second and third pins is the same as the clockwise gap between the first and second pins. The process continues, at each step preserving the clockwise gap between successive pins.

If a pin that's already been used is reached, the current thread is tied off. If there are any unused pins, the process starts over with a different thread, keeping the same clockwise gap between pins. We continue in the same way until all the pins are used up.

We pose the following question: Given the number of pins, and the clockwise gap, how many different threads will be used?

Example: If there are five pins, and we use a clockwise gap of two, then it only takes one thread. If there are six pins, and we use a gap of three, then it takes two threads.

2. In the land of Bork Bork, they have stamps worth 5 Borks and 11 Borks. If it costs 346 Borks to send a letter, is it possible to get exact postage with 5 and 11 Bork stamps? What about 52 Borks? 36 Borks?

What different postage prices are possible using only 5 Bork and 11 Bork stamps?