# LA MATH CIRCLE, OCT 4TH 2009, WORKSHEET

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**Problem 1:** (team effort) Calculate the continued fraction expansion for square roots of small integers that are not integers themselves:

$$\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \ldots, \sqrt{30}$$

(or as far as we get). For each, record the sequence of integer parts

$$m_1, m_2, m_3, \ldots$$

as well as the sequence of remainders

$$r_1, r_2, r_3, \ldots$$

so that

$$\sqrt{n} = m_1 + \frac{1}{r_1} = m_1 + \frac{1}{m_2 + \frac{1}{r_2}} = m_1 + \frac{1}{m_2 + \frac{1}{m_3 + \frac{1}{r_3}}} = \ldots$$

Each term in the sequence $r_1, r_2, \ldots$ should be written as

$$\frac{b\sqrt{n} + a}{c}$$

with integers $a, b, c$. Stop your calculation when you see a pattern emerge.

**Problem 2:** Why can we have $b$ always equal to 1? Why is $a, c$ always positive? Why is always $a < \sqrt{n}$ and $c \leq \sqrt{n} + a$ ?, Why does $c$ always divide $n - a^2$? It helps to assume all these properties for one of the remainders, and show them for the next remainder (That's called induction, provided you show it also for the very first remainder.)

**Problem 3:** Why are the sequences $m_1, m_2, \ldots$ and $r_1, r_2, \ldots$ always eventually periodic?

**Problem 4:** Can you slightly modify Problem 1) so that the periodic pattern in the sequence $m_1, m_2, \ldots$ starts with the very first term? More precisely which numbers should we have asked to calculate the continued fraction of?

**Problem 5:** Do you find any more symmetries of the sequence $m_1, m_2, \ldots$?

**Problem 6:** Can you explain that symmetry by looking at $r_1, r_2, \ldots$?

**Problem 7:** What can you say about the symmetry points of this symmetry in the case when the length of the period is odd?

**Problem 8:** Now assume $n$ is a prime number. What can you say about the possible symmetry points if the period is even?

**Problem 9:** Now assume $n$ is a prime number of the form $4k + 1$ for some integer $k$. Can you prove that $n$ can be written as the sum of two squares: $n = a^2 + b^2$ ?

**Problem 10:** Prove Fermat's Theorem on sums of squares, first proven by Euler: An odd prime number is the sum of two squares if and only if dividing it by 4 gives a remainder of one.