Sum-functions, Mobius inversion, and all that.

May 24, 2009

Constructing new functions out of old ones

We will use 3 standard arithmetic functions defined as follows:

- The delta-function:
$$\delta(n) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$$
- The identity function:
$$\mathrm{id}(n) = n \quad \text{for all } n$$
- The constant function:
$$I(n) = 1 \qquad \text{for all } n.$$

Suppose that we have a function $f : \mathbb{N} \to \mathbb{C}$ and use it to define a new arithmetic function $g : \mathbb{N} \to \mathbb{C}$ using one of the following ways:

$$g(n) = \sum_{i \le n} f(i)$$
$$g(n) = \sum_{d|n} f(i).$$

A natural question is whether we can reverse this process, i.e., can we recover the function $f(n)$ if we know $g(n)$?

1. Let
$$g(n) = \sum_{i \le n} f(i).$$
Find the expression for $f(1),\ f(2),\ f(3), \ldots$ in terms of $g(n)$. Can you conjecture and prove the general formula?

2. Let
$$g(n) = \sum_{d|n} f(d).$$

(a) Let $g(n) = n$. Show that $f(n) = \phi(n)$ (Use the multiplicativity of the Euler's function $\phi(n)$).

(b) Express $f(p)$, $f(p^2)$, $f(p^n)$ and $f(p_1 \cdot p_2)$ in terms of $g(n)$, where $p, p_1 \neq p_2$ are prime numbers.

Can we get the old functions out of new ones?

We can now use the convolution product to recover $f(n)$ from its sum function $F(n) = \sum_{d|n} f(d)$.

1. Express $F(n)$ as the convolution (Dirichlet product) of f with some function.

2. Take $F = f \circ I$ and take the Dirichlet product of both sides with I^{-1} (we don't know yet that it exists, but we will show it soon!). What do you get? What is the expression for f in terms of F?

3. Now let's find an inverse of I under the Dirichlet product. Denote by $\mu = I^{-1}$. Then we must have

$$\begin{aligned} \mu \circ I &= \delta \\ \sum_{d|n} \mu(d) &= \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases}. \end{aligned}$$

   (a) Find $\mu(1)$; Find $\mu(p)$ for a prime p; Find $\mu(p^2)$; Find $\mu(p_1 p_2)$;

   (b) Conjecture and prove the general formula for $\mu(n)$.

It follows from these problems that

$$f = F \circ \mu.$$

From a meeting of the Berkeley Math Circle

1. Let $\{a_0, a_1, \ldots, a_n, \ldots\}$ and $\{b_0, b_1, \ldots, b_n, \ldots\}$ be two sequences. Show that the following relations are equivalent:

$$a_n = \sum_{k=0}^{n} b_k \text{ for all } n \qquad \Longleftrightarrow \qquad b_n = \sum_{k=0}^{n} (-1)^{k+n} a_k \text{ for all } n.$$

2. Suppose that we are given infinitely many tickets, each with one natural number on it. For any $n \in \mathbb{N}$, the number of tickets on which divisors of n are written is exactly n. For example, the divisors of 6, $\{1, 2, 3, 6\}$, are written in some variation on 6 tickets, and no other ticket has these numbers written on it. Prove that any number $n \in \mathbb{N}$ is written on at least one ticket.