

Combinatorics and The Pigeon Hole Principle
UCLA Math Circle, October 28, 2007 (Luminita Vese)

Recall $C_k^n = \frac{n!}{k!(n-k)!}$.

More combinatorics problems

1. Two students must be chosen out of a group of thirty for a mathematical contest. In how many ways can this be done ?
2. Show $C_k^{m+1} = C_k^m + C_{k-1}^m$.
3. How many ways are there to choose a team of three students out of a group of 30 ?
4. In how many ways can one choose 4 colors out of 7 given colors ?
5. One student has 6 math books, and another has 8 books. How many ways are there to exchange 3 books belonging to the first student with three books belonging to the second student ?
6. There are 2 girls and 7 boys in a chess club. A team of four persons must be chosen for a tournament, and there must be at least 1 girl in the team. In how many ways can this be done ?
7. How many ways are there to divide 10 boys into two basketball teams of 5 boys each ?
8. Ten points are marked on a plane so that no three of them are on the same straight line. How many triangles are there with vertices at these points ?
9. Ten points are marked on a straight line, and 11 points are marked on another line, parallel to the first one. How many
 - (a) triangles;
 - (b) quadrilateralsare there with vertices at these points ?
10. A set of 15 different words is given. In how many ways is it possible to choose a subset of no more than 5 words ?
11. There are 4 married couples in a club. How many ways are there to choose a committee of 3 members so that no two spouses are members of the committee ?
12. There are 31 students in a class, including Pete and John. How many ways are there to choose a soccer team (11 players) so that Pete and John are not on the team together ?
13. How many ways are there to rearrange the letters in the word "ASUNDER" so that vowels will be in alphabetical order, as well as consonants ? Example: DANERUS (A-E-U, D-N-R-S).
14. We must choose a 5-member team from 12 girls and 10 boys. How many ways are there to make the choice so that there are no more than 3 boys on the team ?
15. How many ways are there to put 12 white and 12 black checkers on the black squares of a chessboard ?

The Pigeon Hole Principle

If we must put $N + 1$ or more pigeons into N holes, then some pigeon hole must contain two or more pigeons.

The proof of the principle (method of proof by contradiction): Suppose no more than one pigeon were in each hole. Then there would be no more than N pigeons altogether, which contradicts the assumption that we have $N + 1$ pigeons.

1. A bag contains beads of two colors: black and white. What is the smallest number of beads which must be drawn from the bag, without looking, so that among these beads there are two of the same color ?

2. One million pine trees grow in a forest. It is known that no pine tree has more than 600000 pine needles on it. Show that two pine trees in the forest must have the same number of pine needles.

3. Given 12 integers, show that two of them can be chosen whose difference is divisible by 11.

4. The city of Leningrad has five million inhabitants. Show that two of these must have the same number of hairs on their heads, if it is known that no person has more than one million hairs on his or her head.

5. Twenty-five crates of apples are delivered to a store. The apples are of three different sorts, and all the apples in each crate are of the same sort. Show that among these crates there are at least nine containing the same sort of apple.

The General Pigeon Hole Principle

If we must put $Nk + 1$ or more pigeons into N holes, then some pigeon hole must contain at least $k + 1$ pigeons.

Proof: Exercise

- The case $k = 1$ reduces to the simple Pigeon Hole Principle.

7. Given 8 different natural numbers, none greater than 15, show that at least three pairs of them have the same positive difference (the pairs need not be disjoint as sets).

8. Show that in any group of five people, there are two who have an identical number of friends within the group.

14. Show that an equilateral triangle cannot be covered completely by two smaller equilateral triangles.

15. Fifty-one points are scattered inside a square with a side of 1 meter. Prove that some set of three of these points can be covered by a square with side 20 centimeters (1 meter = 100 centimeters).

Another generalization

If the sum of n or more numbers is equal to S , then among these there must be one or more numbers not greater than S/n , and also one or more numbers not less than S/n .

Proof (by contradiction)

If all the numbers are greater than S/n , then their sum would be bigger than S , which contradicts the assumption.

If all the numbers are smaller than S/n , then their sum would be smaller than S , which contradicts again the assumption.

16. Five young workers received as wages 1500 dollars altogether. Each of them wants to buy a cassette player in the amount of 320 dollars. Prove that at least one of them must wait for the next paycheck to make his purchase.

17. In a brigade of 7 people, the sum of their ages of the members is 332 years. Prove that three members can be chosen so that the sum of their ages is no less than 142 years.

19. Prove that there exist two powers of two which differ by a multiple of 1987.

20. Prove that of any 52 integers, two can always be found such that the difference of their squares is divisible by 100.

24. Of 100 people seated at a round table, more than half are men. Prove that there are two men who are seated diametrically opposite to each other.

25. Fifteen boys gathered 100 nuts. Prove that some pair of boys gathered an identical number of nuts.

32. Prove that we can choose a subset of a set of ten integers, such that their sum is divisible by 10.