The Mathematics of Image Processing
The TV Norm and Signals

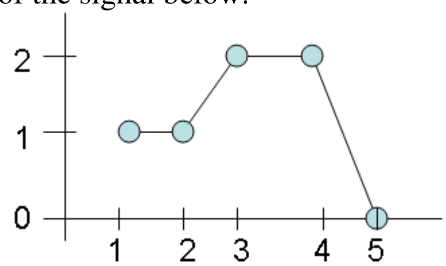
UCLA Math Circle April 5, 2009

The *Total Variation (TV) norm* is a useful statistic for measuring the "quality" of an image. A clean image should have a low TV value, while a noisy image should have a large TV value. We can use this statistic to remove the noise from a corrupted image by creating an image with a lower TV value. We call this process *TV denoising* or *TV minimization*.

For a 1-D discrete signal $f = (f_1, f_2, \ldots, f_N)$, the TV norm is defined as

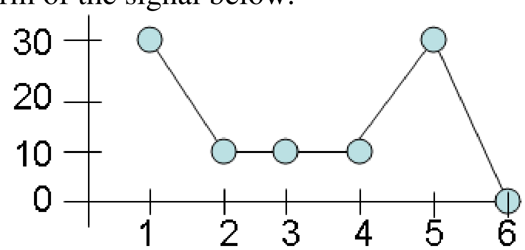
$$TV(f) = \sum_{i=2}^{N} \left| f_i - f_{i-1} \right|$$

<u>Ex</u> Compute the TV norm of the signal below.

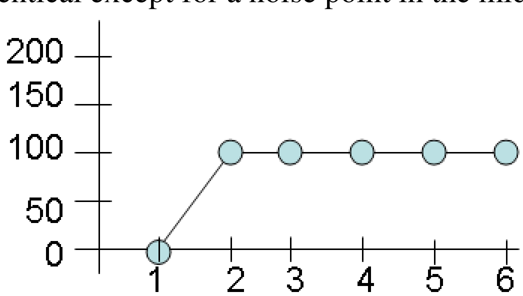


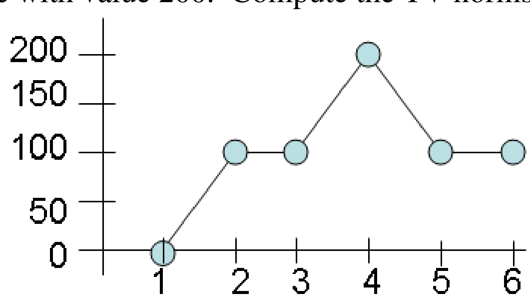
TV = $|f_2-f_1| + |f_3-f_2| + |f_4-f_3| + |f_5-f_4|$ = $|1-1| + |2-1| + |2-2| + |0-2|$ = $0 + 1 + 0 + 2$ = 3

1.) Compute the TV norm of the signal below.

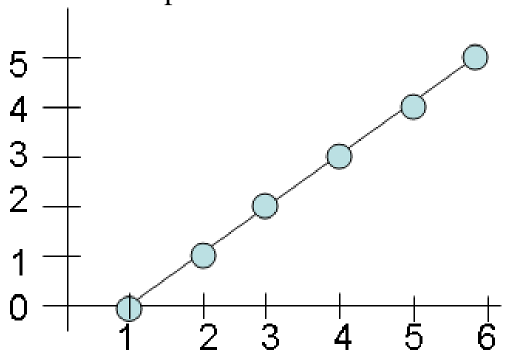


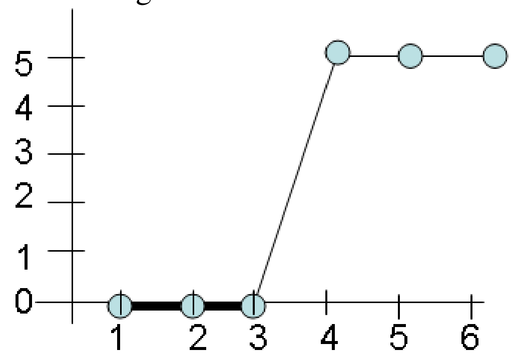
2.) Let's see the effect of adding a noise point to a signal. The two signals below are identical except for a noise point in the middle with value 200. Compute the TV norms.



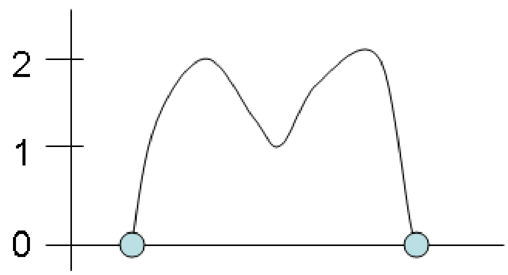


3.) As the name implies, Total Variation counts the total amount of "jump" in the signal. But a key fact about TV is that it counts the jump, but doesn't really care *how* the signal jumped. As an example, the two signals below each jump from 0 to 5 but the signals have different shapes. Calculate the TV value of each signal.





4.) The signal below does not show the values along the x-axis. Can you still calculate the TV norm?



5.) What is the TV norm of the function $f(x) = \sin x$ on the interval $[0,2\pi]$?

6.) What signal would have the minimum TV value: TV=0 ?

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A signal is really a 1-D image. You can think of a signal as being just a really short image that is only 1 pixel high. Now let's extend the TV norm to 2-D so we can compute the TV of images. Let u(x,y) be an MxN image. That is, we have M rows (y-direction) and N columns (x-direction). For 2-D images, the *anisotropic* TV norm is defined as:

$$TV(u) = \sum_{x=2}^{N}\sum_{y=2}^{M} \left|u(x,y) - u(x-1,y)\right| + \left|u(x,y) - u(x,y-1)\right|$$

So at each pixel, we compute the jump with the neighbors and then add them up.

<u>**Ex**</u> An image is show below with its pixel values. Compute the TV norm.

0	0	0	0
0	0	1	3
0	0	1	0

TV = 1 + 1 + 1 + 1 +2 + 3 + 3 = 12

1.) Compute the TV norm of the images below.

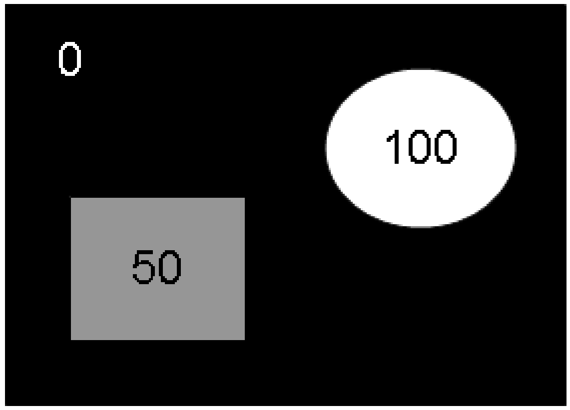
0	0	0	0
0	3	3	0
0	3	3	0
0	0	0	0

1	1	1	1
0	0	1	3
0	0	1	3
1	1	1	3

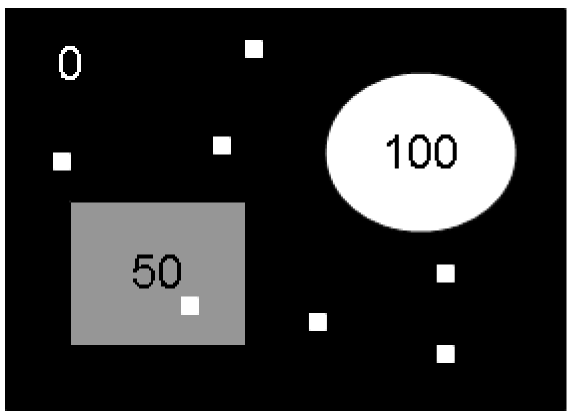
Look at your answer to #1. Hopefully, you notice that the TV value is related to the perimeter of the shapes. There is a beautiful mathematical theorem called the *Co-Area Formula* which says the TV norm of any image is

TV = (Perimeter of each shape) * (Jump along the shape boundary)

2.) Using the Co-Area Formula, calculate the TV norm of the images below. The gray square has size 15x15. The white circle has radius 8 pixels.



3.) Let's see what happens to the TV norm when we add noise. The white dots in the image below have value 100 and are tiny 1x1 squares. Calculate the TV norm.



4.) We call this version of the TV norm *anisotropic* because the orientation of the edge makes a difference in the calculation. The anisotropic TV norm prefers edges that are horizontal or vertical, rather than at an angle. The two images below are both half black and half white. Calculate the TV of the two images below, assuming 0=black and 1=white.