Inequalities in Plane Geometry*
(LAMC 10/07/07)

Olga Radko

October 3, 2007

A lot of facts and theorems of plane geometry are expressed by equalities. Such is the case of the statements:
"The sum of the angles of a triangle equals 180 degrees."
"The square of the hypothenuse of a right angle triangle is equal to the sum of the squares of the other two sides."
However, inequlities relating various geometric quantities sometimes play even more basic and fundamental role.

Today we will concentrate on two important inequalities:

1. ***The Triangle Inequlity:*** *For any three points on the plane the distance between two of the points is no bigger then the sum of the distances from these points to the third one.*
   In symbols, for any three points $A,\ B,\ C$ on the plane we have
   $$|AC| \leq |AB| + |BC|.$$
   Moreover, the equality takes place if and only if the three points are on the same line.
   I.e., $|AC| = |AB| + |BC|$ if and only if $A,\ B,\ C$ lie on the same line.

2. *In a triangle, the larger of any two sides is the side opposite to the larger angle.*
   In symbols, for a triangle ABC we have: $|AB| > |AC|$, if and only if $\angle C > \angle B$.

*This session is based on the material from the book "Mathematical Circles (Russian Experience)" by D. Fomin, S. Genkin, I. Itenberg.

Introductory problems

1. Prove that for any three points A, B, C on the plane we have $|AB| \geq |AB - BC|$.

2. Show that the length of any side of a triangle is not more than half its perimeter.

3. In triangle ABC the length of AC is 3.8 cm, the length of AB is 0.6. If the length of side BC is an integer, what is this length?

4. Let $ABCD$ be convex quadrilateral. For any point P inside of the quadrilateral consider the sum of the distances from P to all the vertices, i.e., consider
$$|PA| + |PB| + |PC| + |PD|.$$
Show that the point for which this sum is minimal is the point of intersection of the diagonals.

5. (Compare with problem 4) Show that the sum of the diagonals of a convex quadrilateral is less than the perimeter but less then double the perimeter.

6. Show that the sum of diagonals of a convex pentagon is greater than the perimeter but less than double the perimeter.

7. Let $ABCD$ be a square and O be a point on the plane. Show that
$$|OA| \leq |OB| + |OC| + |OD|.$$

8. Prove that if you can orm a triangle from segments with lengths a, b, c, then you can do this with segments with lengths $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$.

9. In triangle ABC the median AM is longer than half of BC. Prove that angle BAC is acute.

10. Let $ABCD$ and $A_1B_1C_1D_1$ be two convex quadrilaterals whose corresponding sides are equal. Prove that if $\angle A > \angle A_1$, then $\angle B < \angle B_1$, $\angle C > \angle C_1$ and $\angle D < \angle D_1$.

11. Prove that the median of a triangle whch lies between two of its unequal sides forms a greater angle with the smaller of those sides.

Problems on triangle inequalities and geometric transformations

Very often the triangle to which we must apply the triangle inequality does not appear in the diagram for the problem. In these cases, a suitable choice of geometric transformation can help.

1. Let l be a horizontal straight line. Let A and B be two arbitrary points in the upper-semiplane. Find point C on the line l so that the sum of the distances $|AC| + |CB|$ is minimal possible. (You might think of the following story that goes with the picture: the line represents the bank of a river, and A and B are initial and final destinations of a traveler who has to take some water from the river on the way from A to B, but would like to minimize the total distance he travels).

2. Let A be a point inside of an acute angle. Find two points B and C on different sides of the angle such that the perimeter of the triangle ABC is the minimal possible. (You can think of the angle as a peninsula, and of point A as a woodsman's hut. The woodsman wants to walk to one shore, then to the other shore and then come back to his hut so that the total distance he travels is minimal.)

Problems involving "unfolding" a surface

1. A fly sits on one vertex of a wooden cube with side length 1. What is the shortest path it can follow to the opposite vertex?

2. A fly sits on the outside surface of a cylindrical drinking glass. It must crawl to another point, situated on the inside surface of the glass. Find the shortest pass possible (neglecting the thickness of the glass).