Cyclic Quadrilaterals

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1 Motivation

Any three non-collinear points always lie on some common circle, but in general it is rare for four points to have that property—unless one is trying to solve an Olympiad problem. Then, the observation that certain quadrilaterals are cyclic often turns out to be the key to the solution. We will survey some facts about cyclic quadrilaterals, and use them to solve actual Olympiad problems.

2 Warm-Up

1. Let ABC be a triangle. Show that one can always find a circle through all three vertices, and prove that the circle is unique.
2. Find a quadrilateral $ABCD$ with the property that no circle passes through all four vertices.
3. Let $ABCD$ be a cyclic quadrilateral. Why is $\angle ABD = \angle ACD$?
4. Let $ABCD$ be a cyclic quadrilateral. Prove that $\angle A + \angle C = 180°$.
5. Let $ABCD$ be a cyclic quadrilaterals, and let its diagonals AC and BD intersect at P. Show that $AX \cdot XC = BX \cdot XD$.

3 Recognizing cyclic quadrilaterals

- A quadrilateral $ABCD$ is cyclic if and only if $\angle ABD = \angle ACD$.
- A quadrilateral is cyclic if and only if its opposite angles sum to 180°.
- Let $ABCD$ be a quadrilateral, and let its diagonals AC and BD intersect at X. Then it is cyclic if and only if $AX \cdot XC = BX \cdot XD$.
- A quadrilateral is cyclic if the problem says it is.
- But if the problem doesn't say a quadrilateral is cyclic, it might still be cyclic.
- And even if the problem doesn't seem to have any quadrilaterals at all, there might be a cyclic one.

4 Problems

1. (Classical) Let ABC be a triangle, and let D, E, F be the feet of the altitudes from A, B, C, respectively. How many sets of 4 cyclic points can you find?
2. (USAMO 1990/5) An acute-angled triangle ABC is given in the plane. The circle with diameter AB intersects altitude CC' and its extension at points M and N, and the circle with diameter AC intersects altitude BB' and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle.

3. (Simson Line) Let ABC be a triangle, and let M be a point on its circumcircle. Let D, E, F be the feet of the perpendiculars from M to BC, CA, AB. Prove that D, E, F are collinear.

4. (R. Gelca 1997) Let A, B, C be collinear and $M \notin AB$. Prove that M and the circumcenters of MAB, MBC, and MAC lie on a circle. **Note:** Inversion, followed by a homothety of ratio $1/2$ about M, reduces this to the previous problem. You are welcome to try to find an alternative solution, however.

5. (R. Gelca 1997) In a circle, AB and CD are orthogonal diameters. A variable line passing through C intersects AB at M and the circle at N. Find the locus of the intersection of the parallel to CD through M with the tangent at N.

6. (Half of USAMO 1993/2) Let $ABCD$ be a convex quadrilateral whose diagonals are orthogonal, and let P be the intersection of the diagonals. Let E, F, G, H be the feet of the perpendiculars from P to AB, BC, CD, DA.

   (i) Prove that $\angle PGF = \angle ACB$.

   (ii) Prove that E, F, G, H are cyclic.

7. (Full USAMO 1993/2) Let $ABCD$ be a convex quadrilateral whose diagonals are orthogonal, and let P be the intersection of the diagonals. Prove that the four points that are symmetric to P with respect to the sides form a cyclic quadrilateral.

8. (R. Gelca 1997) Let B and C be the endpoints, and A the midpoint, of a semicircle. Let M be a point on the line segment AC and $P, Q \in BM$, with $AP \perp BM$ and $CQ \perp BM$. Prove that $BP = PQ + QC$.

9. (R. Gelca 1998) Let $ABCD$ be a cyclic quadrilateral with $AC \perp BD$. Prove that the area of quadrilaterals $AOCD$ and $AOCB$ are equal, where O is the circumcenter.