Induction

November 13, 2008

Motivational problems:

1. (Warm-up) How many grandparents did all of your grandparents have all together?

2. Continue the sequence:

$$11,\ 21,\ 41,\ 81,\ 161,\ ...$$

3. Look at the following sums:

$$1,\ 1+3,\ 1+3+5,\ 1+3+5+7,\ ...$$

   Do you think there is a simple rule for the values of these sums?

4. Look at the following sums:

$$1,\ 1+8,\ 1+8+27,\ 1+8+27+64,\dots$$

   Do you think there is a simple rule for the values of these sums?

5. One square (box) was cut off from a 16×16 square graph paper. Prove that the figure obtained can be dissected into trominos of a certain type — "corners". (*Hint:* can you cut a much smaller square into the trominos? Can you build up on this to increase the size of the square?)

6. Can you find 10 integers, such that their sum is divisible by each of them. (*Hint:* start with finding 3 numbers whose sum is divisible by each of them).

7. Prove that the number 111....11 (written with 3^5 digits 1 is divisible by 3^5). (*Hint* : can you find an analogous statement for smaller numbers? Can you prove it?)

8. Is it true that the number n^2+n+41 is prime for any natural number n? (Recall that an integer number is *prime* if it is divisible only by 1 and by itself).

What is induction?

The Method of Mathematical Induction (MMI) is usually stated as one of the axioms of the natural numbers (so-called Peano axioms), and thus, does not require a proof.

Let $P(n)$ be a mathematical statement that depends on an integer n. Examples of such statements include identities, inequalities, statements about divisibility of numbers. E.g.:

- $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$;
- $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9;
- $2^n > n$.

One should think of $P(n)$ not as a single statement but as an infinite series of similar propositions: $P(1),\ P(2),\ P(3), \ldots$ for all integer values of n.

Principle of Mathematical Induction:
Suppose that

1. $P(1)$ *is true*
2. *For any* $n \geq 1$ *"*$P(n)$ *is true" implies "*$P(n+1)$ *is true".*

Then $P(n)$ *is true for all* n.

In our analogy with an (infinite!) row of dominos, this can be remembered as follows:
Suppose that

1. *We can knock down the first domino;*
2. *The dominos are so close, that each previous will knock the following one down when falling.*

Then all the dominos will be down.

To prove a statement using the Method of Math Induction, we need to complete two steps:

1. *Base case*: show that the statement $P(1)$ is true.
2. *Inductive step:* Assume that $P(n)$ is true for some n. Show that this implies that $P(n+1)$ is also true.

A paradox: "All the horses are of the same color!"

The following example was invented by George Polya to demonstrate that one has to be careful when applying the MMI.

Theorem. All horses are of the same color.

(Similarly, you can "proof" that, e.g., "all people are of the same height" or that "all test problems are equally difficult", etc.)

"Proof" (by induction).

1. *The base case:* consider a set consisting of one horse. In this set all horses are clearly of the same color (as there is just one horse).

2. *Inductive step:* Assume that the statement is true for n horses. (That is, in any set of n horses all horses are of the same color). Consider a set consisting of $(n+1)$ horse. Enumerate the horses in this set by number $1, \ldots, n+1$. Consider the following two sets of horses: $(1, 2, \ldots, n-1, n)$ and $(2, 3, \ldots, n, n+1)$. Each of the sets consists of n horses. By induction assumption, in each of the sets all the horses are of the same color. Since the two sets overlap, all the horses are of the same color.

The argument above obviously contains a mistake. Can you find it?

Another example:

1. STATEMENT: there is a unique line going through any $n \geq 2$ points on the plane.
   "PROOF": We will apply math induction.
   Base case ($n = 2$): By axioms of Geometry, there is a unique line going through any 2 points.
   Inductive step: Assume that we can draw a unique line through n points. Now suppose that we have $n+1$ points. By assumption, there is a unique line through n points $1, 2, \ldots, n$ and a unique line through n points $2, \ldots, n+1$. These lines are the same and contain all $n+1$ points. Thus, there is a line going through all $n+1$ points.
   This concludes proof by induction.
   Find a mistake in this "proof".

More problems:

1. "Tower of Hanoi" game: There are 3 spindles on a base, with n rings on one of them. The rings are arranged in order of their size (the largests is on the bottom). One can move the highest (smallest) ring on any spindle onto another spindle provided that you never put a larger ring on top of a smaller one. Prove that one can move all the rings to one of the free spindles using $2^n - 1$.

2. Show that for any n the following holds:

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

3. Show that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for any n.

4. Show that $4^n + 15n - 1$ is divisible by 9.

5. Prove that the absolute value of the sum of several numbers does not exceed the sum of the absolute values of these numbers.

6. Prove that any natural number can be represented as a sum of several distinct powers of 2.

7. Show that $2^n > n$ for any n.