INVERSION WITH RESPECT TO A CIRCLE

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1. Transformations of the plane

Exercise 1. Recall at least three different transformations of the plane.

(1)

(2)

(3)

Exercise 2. Give an example of a transformation φ and a geometric shape F on the plane such that F is invariant under φ, but none of the points of F is a fixed point of φ.

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2. Symmetry and reflection with respect to a line

Fix a line l on the plane. Reflection with respect to l is a transformation of the plane sending each point A into its mirror image A' with respect to l.

Problem 1. *Let A and B be two points on the plane. Find the set of all points which are equidistant from A and B. I.e., find the set of all points X on the plane which satisfy*

$$|AX| = |XB|. \tag{1}$$

Note that you need to prove two things:

1. *All the points in your set satisfy this condition.*
2. *No other points satisfy this condition.*

(The answer is closely related to the reflection with respect to a line)

(1)

(2)

3. Two problems

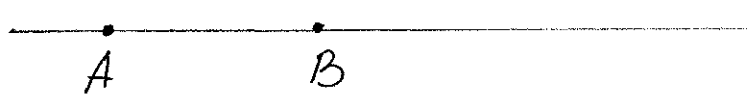
Apollonian families. An ancient Greek mathematician Apollonius (260 BC - 190 BC) proposed the following generalization of Problem 1:

Problem. *(Apollonius) Let A and B be two points on the plane, and let k be a positive number. Find the set of all points X satisfying*

$$|AX| = k \cdot |XB|. \tag{2}$$

Consider the following:

(1) *Let $k = 2$. How many points X on the line AB satisfy (2)? Mark these points. Find several other points X (not lying on the line AB) such that the condition (2) is satisfied. Can you make a guess how the solution of (2) looks like for $k = 2$? (make a sketch)*



(2) *Repeat the same for $k = 3$.*

(3) Explain why it is enough to consider the problem only for $k > 1$. We will be able to solve this problem completely using Inversion.

A construction problem. Apollonius also proposed the following problem:

Given three circles, construct a circle tangent to all of them.

Comments:

(1) Two circles are *tangent* to each other if they have exactly one point in common.
(2) Depending on the mutual positions of the given circles there could be one, many, or no solutions.
(3) For ancient Greeks, "construct" meant "construct using a ruler and a straight edge".
(4) Circles here can be understood in the very general sense. In particular, they can be
  - points (circles of radius 0);
  - lines (circles of infinite radius);

A solution of this construction problem (and of many others!) is ~~a~~ based on learning about the same transformation called Inversion.

Exercise 3. Sketch an example of three circle such that

(1) Apollonius problem has no solutions:

(2) Apollonian problem has two solutions:

(3) Apollonius problem has infinitely many solutions:

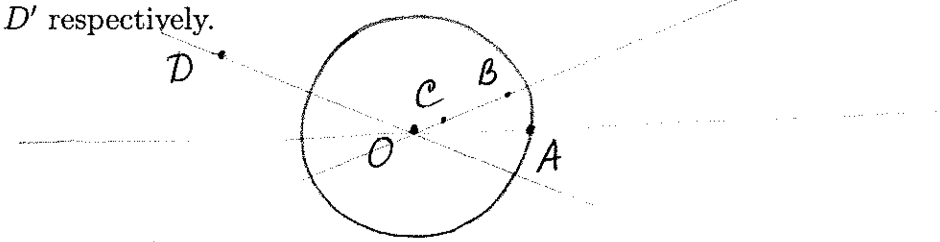
4. Inversion (with respect to a circle)

Fix a circle of radius R with center at a point O on the plane. *Inversion with respect to this circle*[1] is the following transformation:

For any point $A \neq O$ on the plane, its image is the point $A' = I(A)$ such that

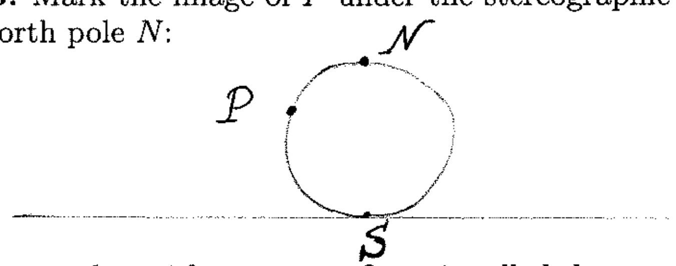
(1) *A' lies on the ray OA.*
(2) $|AO| \cdot |OA'| = R^2$.

Exercise 4. Mark (approximately) the images of points A, B, C, D under inversion with respect to the circle below. Denote the images by A', B', C', D' respectively.



To define inversion for O, recall the stereographic projection. Let S be a unit sphere lying on a (horizontal) plane. Let N be the North pole of the sphere. For any point $P \neq N$ on the sphere there is a unique line going through N and P. The image of P under stereographic projection is the point of intersection of this line with the plane. Clearly, this establishes a one-to-one correspondence of points on the plane with points on the sphere with the exception of the North pole. Now add to the plane an artificial *"point at infinity"* O_∞ as the point corresponding to the North pole under the stereographic projection.

Exercise 5. Mark the image of P under the stereographic projection from the North pole N:



The plane together with point at infinity is called the *extended plane*. We can now define inversion on the extended plane. Let

$$I(O) = O_\infty, \quad I(O_\infty) = O.$$

Considering the extended plane allows us to view straight line as generalized circles.

[1]If you are familiar with complex numbers, explain why the map $z \mapsto \frac{1}{z}$ is the inversion with respect to the circle $|z| = 1$.

From now on I is the inversion with respect to the circle of radius R with center O.

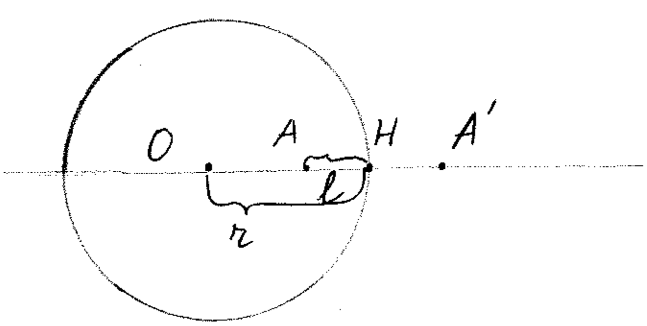
Problem 2. Show that inversion is a kind of *Symmetry*:

(1) Points on the circle of inversion stay fixed. Points inside of the circle of inversion are moved outside. Points outside of the circle of inversion are moved inside.
(2) If A' is the image of A, then A is the image of A'. (In short, $(A')' = A$).

Problem 3. Show that in the limit $r \to \infty$ (i.e., the circle of inversion becomes a line) inversion becomes the reflection with respect to this line.

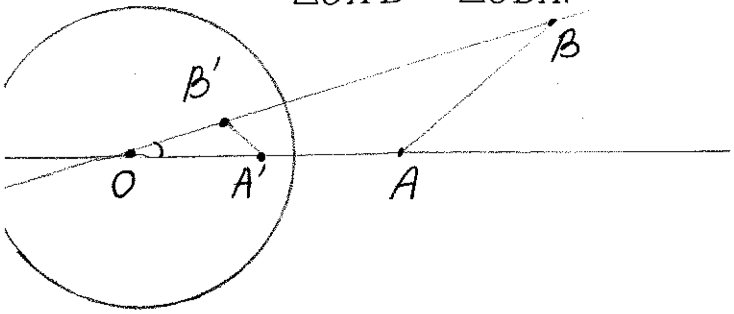
More precisely, let r be the radius of the circle of inversion. Let $|AH| = l$ be the distance from a point A to the circle of inversion.

(1) Compute $|A'H|$ (the distance from the image to the same circle) in terms of r and l.
(2) Show that if $r \to \infty$ then $|A'H| \to l = |AH|$ (i.e., inversion becomes a reflection).



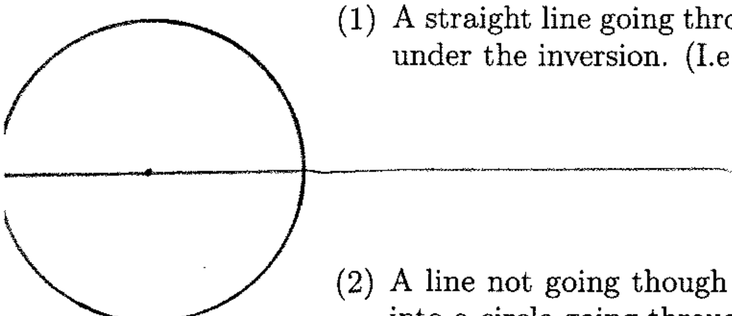
Prove the following useful

Lemma 1. *Let A' and B' be the images of points A and B under inversion with center O with respect to a circle of radius r. Then* $\triangle OA'B' \sim \triangle OBA$.

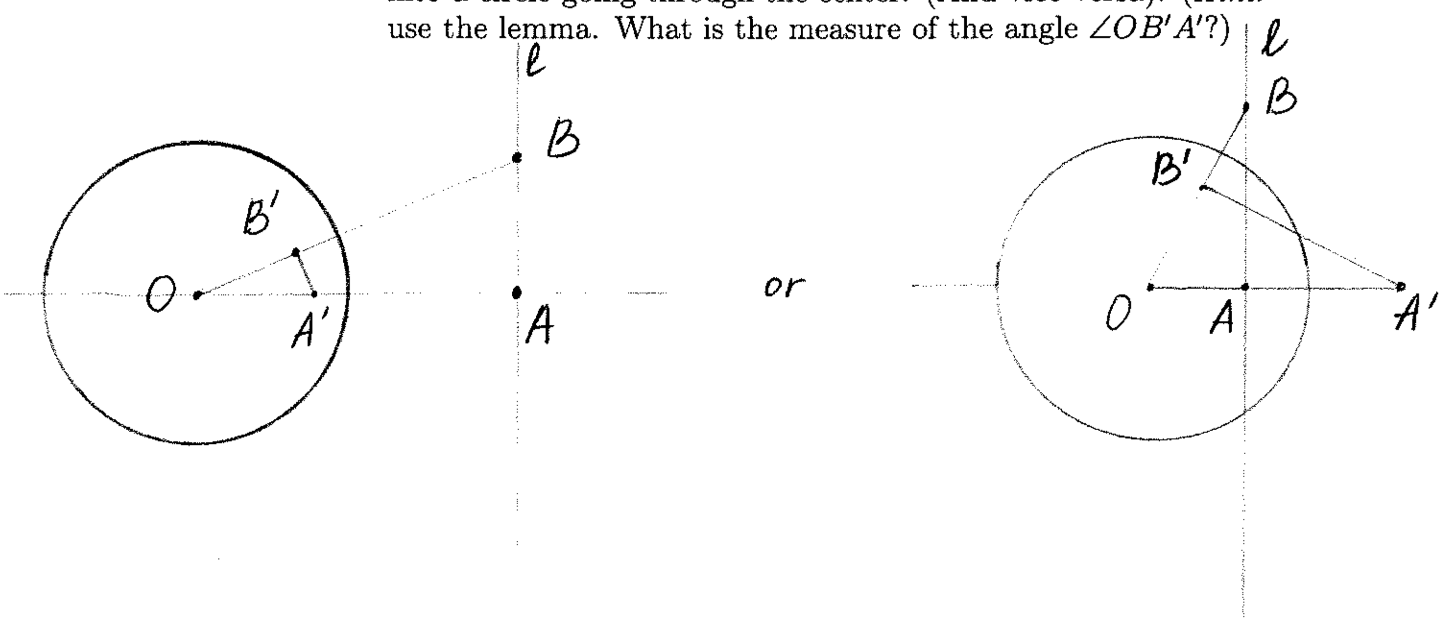


Problem 4. Prove further properties of lines and circles under inversion:

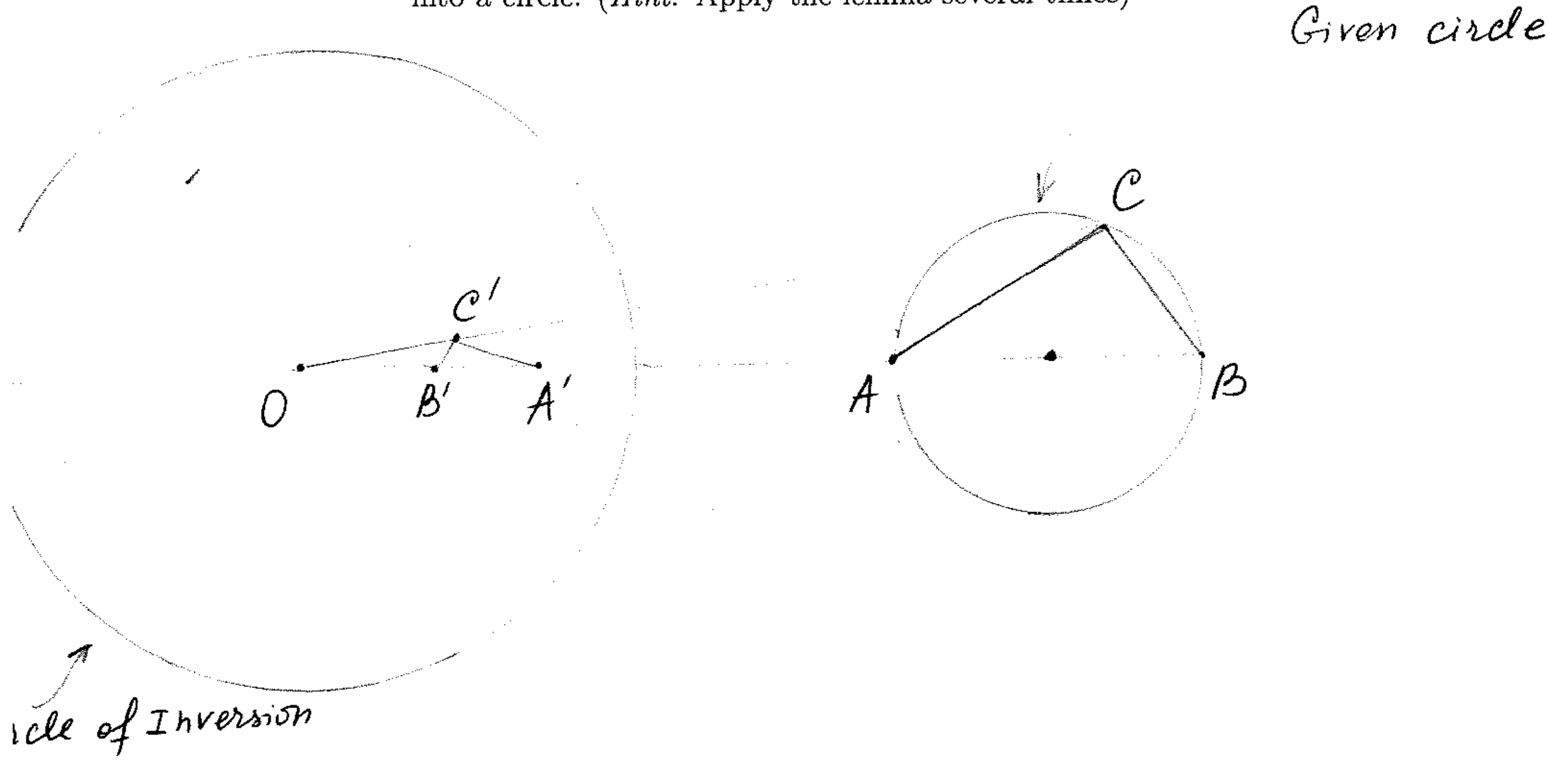
(1) A straight line going through the center of inversion is invariant under the inversion. (I.e., $I(l) = l$ if $O \in l$).



(2) A line not going though the center of inversion is transformed into a circle going through the center. (And vice versa). (*Hint:* use the lemma. What is the measure of the angle $\angle OB'A'$?)



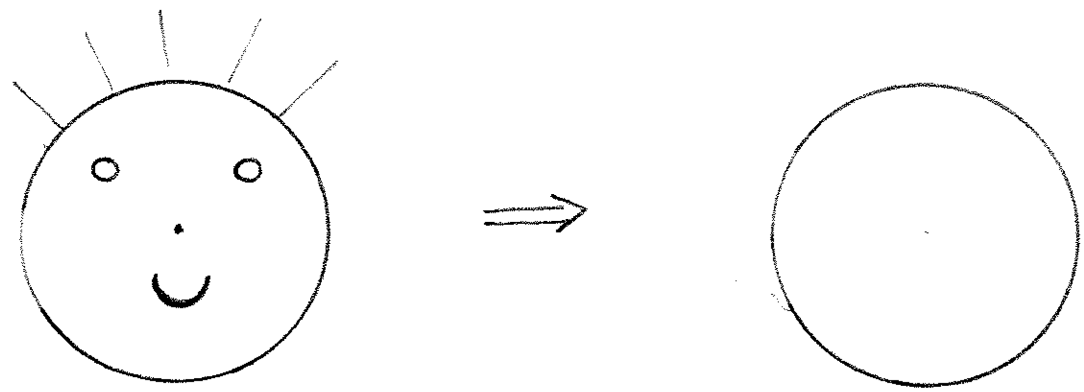
(3) A circle not going through the center of inversion is transformed into a circle. (*Hint:* Apply the lemma several times)

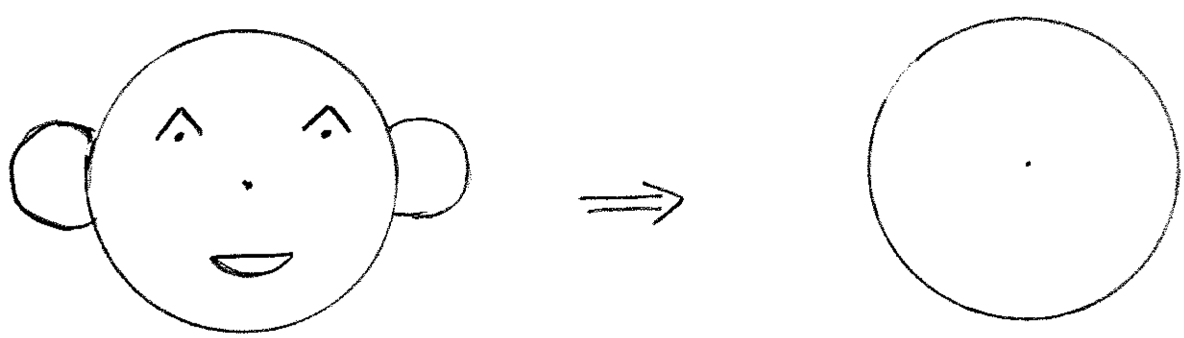


Considering a straight line as a circle of infinite radius containing the point at infinity, we can summarize these properties of inversion as follows:

Inversion transforms (generalized) circles into (generalized) circles. The circles going through the center of inversion and the circles containing the point at infinity are transformed into each other.

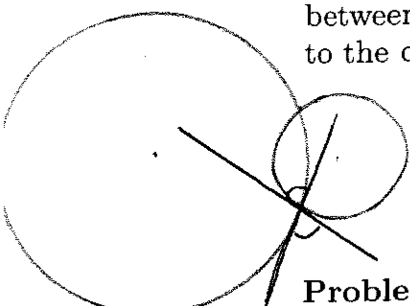
Problem 5. Draw the images of the pictures below under the inversion with respect to the given circle. (*Hint:* Use the properties of inversion that you have obtained earlier):

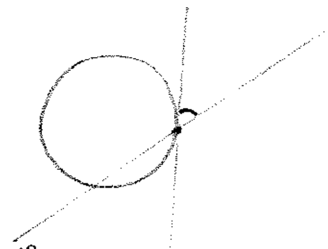
1. 

2. 

Next, we will see how inversion changes the angles between lines and circles.

Definition 1. The angle between two intersecting circles is the angle between their tangent lines at the point of intersection. The angle between a line and a circle is the angle between the line and the tangent to the circle at the point of intersection:



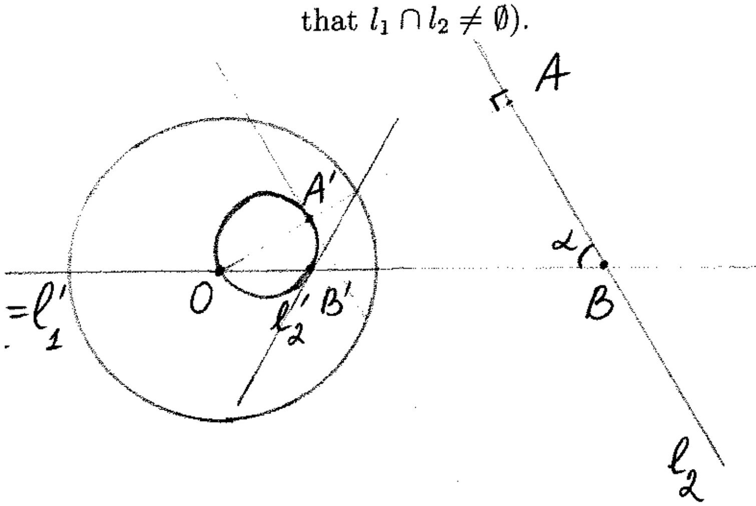


Problem 6. *(Angle-preserving properties)*[2]

Show that the angle between two lines is preserved under inversion. Let l_1 and l_2 be two lines. There are three possible cases:

(1) Both l_1 and l_2 go through O (the center of inversion).
(2) l_1 goes through O, but l_2 does not. (there are two subcases: $l_1 || l_2$ and $l_1 \cap l_2 \neq \emptyset$).
(3) Neither l_1 nor l_2 go through O.

The first case is trivial. Prove the second one (under the assumption that $l_1 \cap l_2 \neq \emptyset$).



The rest of the cases can be considered analogously.

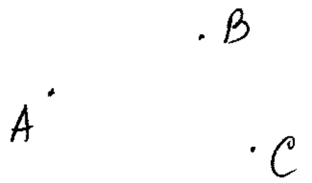
More generally, one can show that inversion preserves the angles between any two (generalized) circles.

[2]The maps which preserve angles are usually called *conformal*.

5. Problems involving inversion

Use properties of inversion to solve the following problems.

Problem 7. Show that for any three points A, B, C on the plane there is a unique circle going through these points. (*Hint:* Use inversion with respect to a circle with center A)

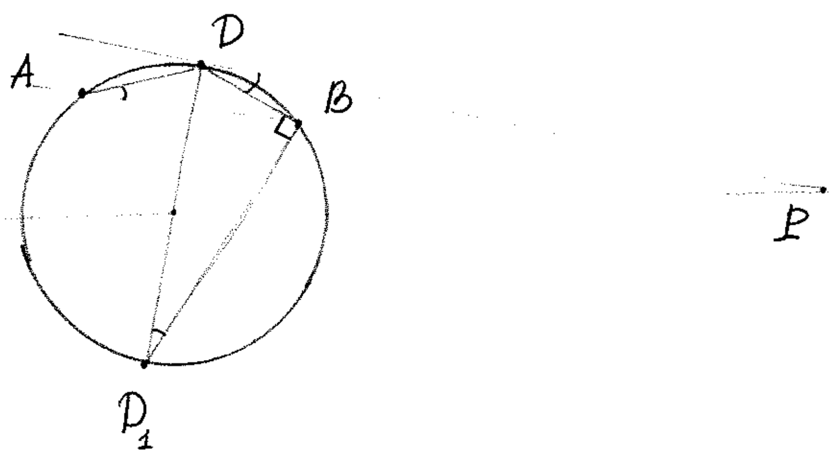


Problem 8. (*The theorem about the square of the tangent*)

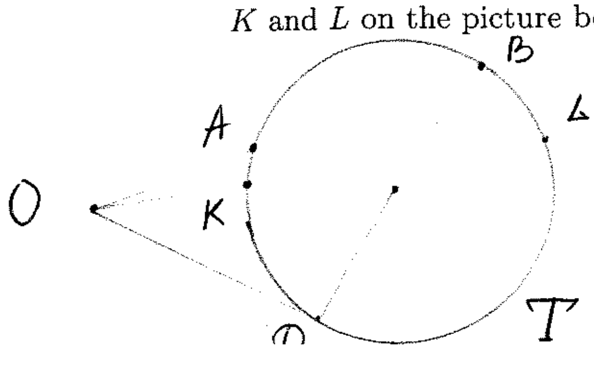
Let P be a point outside of a circle T. Let PD be tangent to the circle, and let PA be a line that intersects the circle at points A and B. Show that

$$|PD|^2 = |PA| \cdot |PB|.$$

(*Hint:* Let O be the center of the circle. Let DD_1 be the diameter of the circle going through D. Use the fact that $\angle DAB = \angle DD_1B$ and that $\angle D_1BD = 90^0$ to show that $\angle DAP = \angle PDB$. After that, prove the statement of the theorem.)



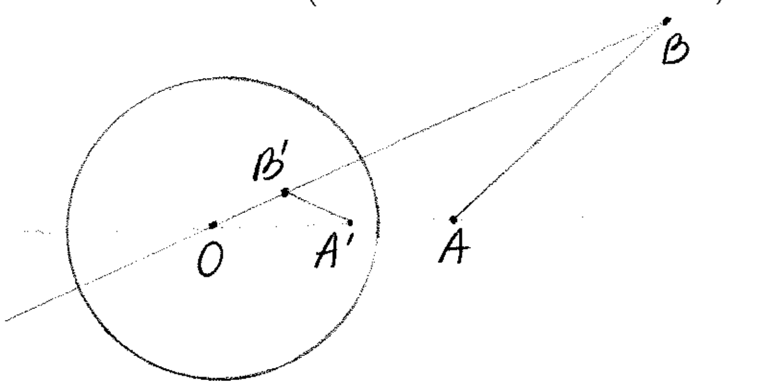
Problem 9. Let A and B be two points on a circle T. Show that if A and B are the images of each other under the inversion with respect to a circle S, then the entire circle T is fixed. (*Hint:* Show that points K and L on the picture below are transformed into each other).



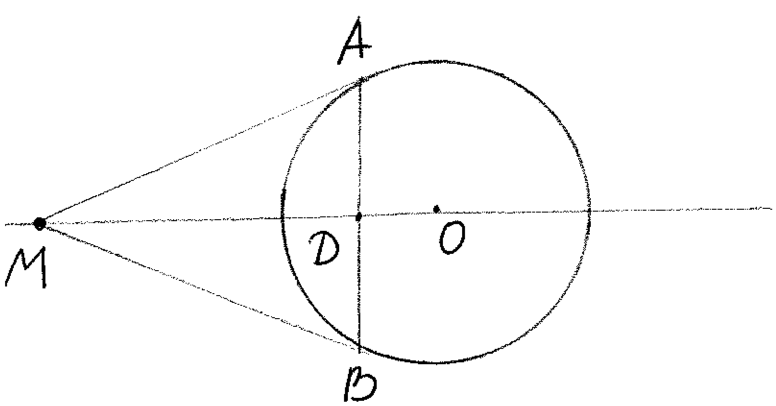
Problem 10. *(Change of distances under inversion)* Let A' and B' be the images of A and B under inversion with respect to a circle with radius R and with center O. Show that

$$|A'B'| = \frac{R^2}{|OA| \cdot |OB|} \cdot |AB| \tag{3}$$

(*Hint:* Use the Lemma above).



Problem 11. Let MA and MB be two lines going through a point M and tangent to a circle S at the points A and B. Let D be the center of the chord AB. Show that D and M are the images of each other under the inversion with respect to S. (*Hint:* Find a pair of similar triangles).

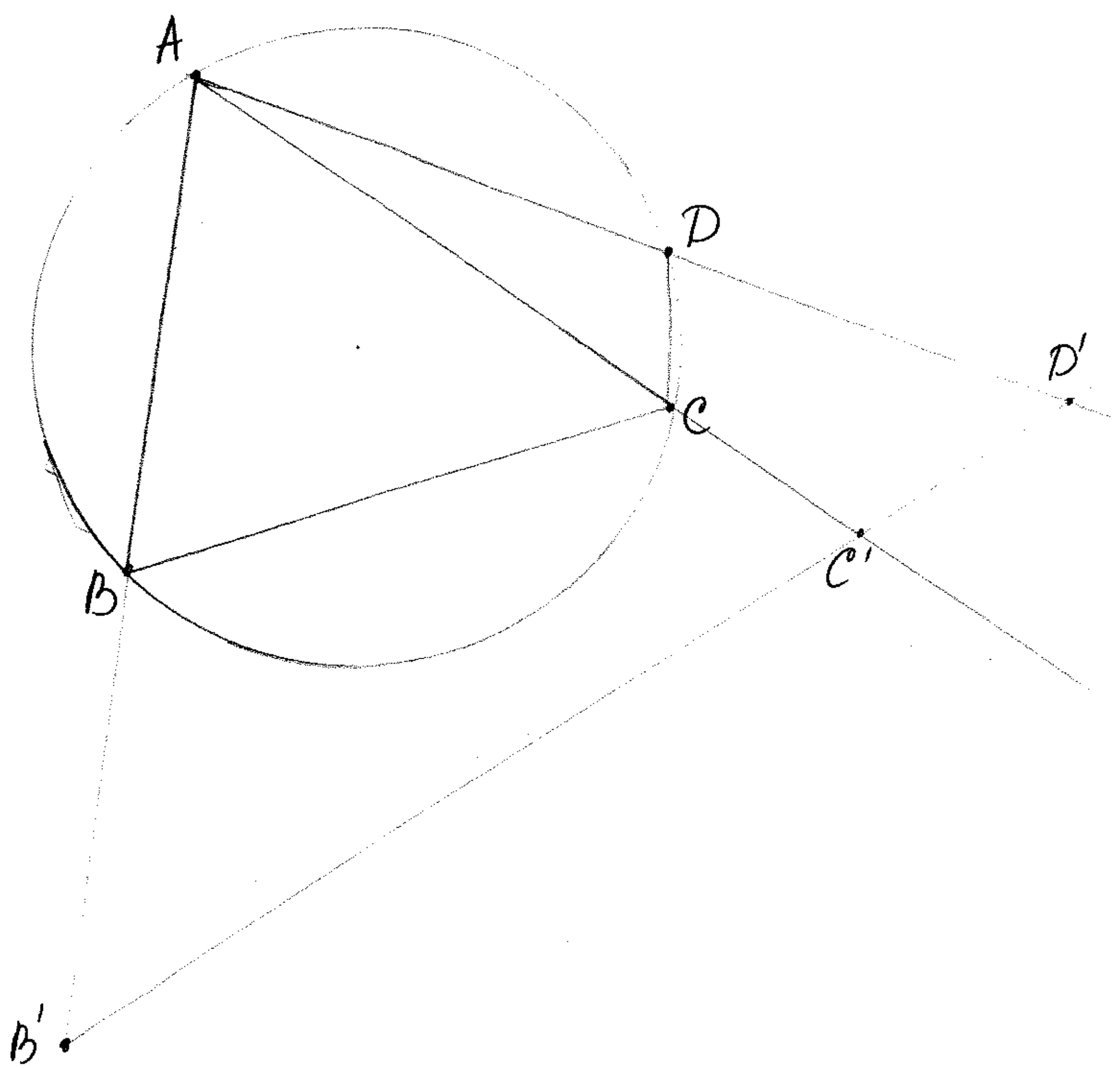


Problem 12. (Ptolemy's theorem)

Let $ABCD$ be an inscribed quadrilateral. Show that

$$|AC| \cdot |BD| = |AB| \cdot |CD| + |BC| \cdot |AD|. \tag{4}$$

(*Hint:* use inversion with center A. Then the circumscribed circle S is transformed into a line l; the vertices B, C, D are transformed into points B', C', D' lying on l. After that use the obvious fact that $|B'D'| = |B'C'| + |C'D'|$ together with the formula for change of distance (3) to get the Ptolemy's formula (4).

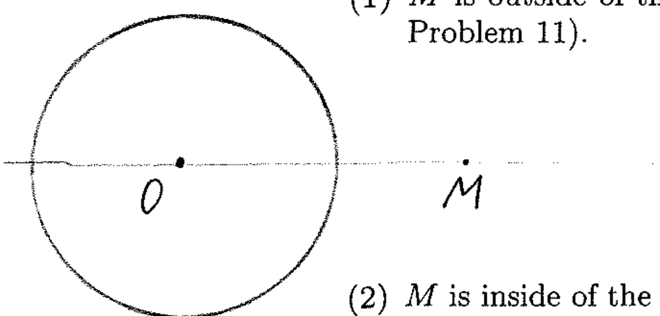


6. Construction problems involving Inversion

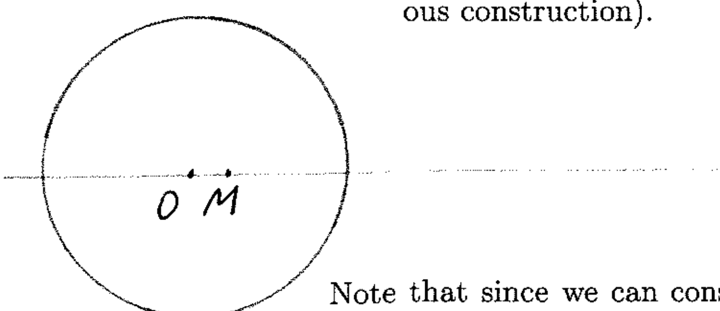
First, we need to be able to construct an image of any point under inversion:

Problem 13. Let M be a point on the plane. Construct its image under the inversion if

(1) M is outside of the circle of inversion (*Hint:* use the result of Problem 11).



(2) M is inside of the circle of inversion (*Hint:* "reverse" the previous construction).



Note that since we can construct the image of any point under inversion, we can also construct the image of any straight line and of any circle. (It's enough to construct the images of any three points lying on the line/circle, and then draw a circle/straight line going through these three points).

One can use the following **Strategy** when solving construction problems using Inversion:

(1) Using inversion, transform a given construction problem into a new (easier!) one;
(2) Solve the new problem.
(3) Using the same inversion, transform the solution to get the solution of the original problem.

Problem 14. Given a line l and two points, A and B, construct a circle S going through A, B and tangent to l. (*Hint:* Assume that such a circle T is constructed. Perform an inversion with center A. What are the images of l and S under this inversion? Use this inversion to solve the original problem).

Problem 15. Use inversion to solve Apollonius problem in the cases that at least two of the circles are tangent to each other:

Let S, T, Q be three circles on the plane such that S and T are tangent to each other. Let O be the point of tangency. Construct a circle tangent to S, T, Q. (*Hint:* an inversion with center O transforms the circles S and T into a pair of parallel lines S' and T'. The third circle, Q, is transformed into a circle or a straight line Q'. Construct the circle tangent to S', T', Q'. Its image under the same inversion is the required circle).

One can show that several other cases can be reduced to this one.

Let's use inversion to solve the problem posed in the beginning: find the set of all points X such that $|AX| = k \cdot |BX|$, where A and B are given fixed points on the plane, and k is a given fixed number.

Problem 16. Let A and B be two fixed points. Consider the set of all points X such that $|AX| = k \cdot |BX|$, where k is a fixed positive number. Let X' be the image of X under inversion with respect to a circle with center A. Let B' be the image of B with respect to the same inversion. Describe the set of all points X'. (*Hint:* assume for simplicity that the circle of inversion has unit radius. Compute the distance $|BX'|$. What does this tell you about the set of all points X'? Perform the same inversion again and describe the set of all points X).