CHEVA'S AND MENELAUS' THEOREMS VIA POINT MASSES

Theorem. *(Cheva, 1678)*

Let A_1, B_1, C_1 be points on the sides (or the extensions of the sides) $BC,\ AC,\ AB$ of $\triangle ABC$ respectively. Then the condition

$$\frac{\overrightarrow{BA_1}}{\overrightarrow{A_1C}} \cdot \frac{\overrightarrow{CB_1}}{\overrightarrow{B_1A}} \cdot \frac{\overrightarrow{AC_1}}{\overrightarrow{C_1B}} = 1$$

implies that either $AA_1,\ BB_1,\ CC_1$ intersect at one point, or they are parallel.

Proof. Let

$$\alpha = \frac{\overrightarrow{AC_1}}{\overrightarrow{C_1B}},\quad \beta = \frac{\overrightarrow{BA_1}}{\overrightarrow{A_1C}},\ \gamma = \frac{\overrightarrow{CB_1}}{\overrightarrow{B_1A}}.$$

Then $\alpha \neq -1,\quad \beta \neq -1,\quad \gamma \neq -1$. Cheva's condition means that

$$\alpha \cdot \beta \cdot \gamma = 1.$$

Place masses 1 at A, α at B and $\alpha\beta$ at C so that

$$\begin{aligned} (1+\alpha)C_1 &= Z(1A,\ \alpha B) \\ (\alpha+\alpha\beta)A_1 &= Z(\alpha B,\ \alpha\beta C) \\ (\alpha\beta+1)B_1 &= Z(\alpha\beta C,\ 1A). \end{aligned}$$

Assume that $1+\alpha+\alpha\beta \neq 0$, so that the system of three masses $1A$, αB and $\alpha\beta C$ have total non-zero mass Z and the center of mass is well-defined. Then

$$\begin{aligned} (1+\alpha+\alpha\beta)\cdot Z(1A, \alpha B, \alpha\beta C) &= \\ &= (1A+\alpha B) + \alpha\beta \cdot C = \\ &= (1+\alpha)C_1 + \alpha\beta C \end{aligned}$$

Therefore, the center of mass lies on CC_1, i.e., $Z \in CC_1$. Similarly, one can prove that $Z \in AA_1$ and $Z \in BB_1$.

Consider the case that $1+\alpha+\alpha\beta = 0$. Since

$$B_1 = \frac{Z(1A,\ \alpha\beta C)}{1+\alpha\beta},$$

we have

$$(1+\alpha\beta)\overrightarrow{BB_1} = 1\cdot\overrightarrow{BA} + \alpha\beta\cdot\overrightarrow{BC}.$$

Similarly,

$$\begin{aligned} (1+\alpha)\overrightarrow{CC_1} &= \\ &= 1\cdot\overrightarrow{CA} + \alpha\overrightarrow{CB} = \\ &= 1\cdot(\overrightarrow{CB}+\overrightarrow{BA}) + \alpha\overrightarrow{CB} = \\ &= 1\cdot\overrightarrow{BA} - (1+\alpha)\overrightarrow{BC} \end{aligned}$$

Since $1+\alpha = -\alpha\beta$, it follows that $\overrightarrow{BB_1} \parallel \overrightarrow{CC_1}$. Similarly, $CC_1 \parallel AA_1$. $\square$

Theorem. *(Menelaus theorem) If A_1, B_1, C_1 are points on the sides (or the extensions of the sides) BC, AC and AB respectively, s.t.*

$$\frac{\overrightarrow{BA_1}}{\overrightarrow{A_{1C}}} \cdot \frac{\overrightarrow{CB_1}}{\overrightarrow{B_1A}} \cdot \frac{\overrightarrow{AC_1}}{\overrightarrow{C_1B}} = -1,$$

the points A_1, B_1 and C_1 lie on the same line.

Proof. Let α, β, γ be as in the proff of Cheva's theorem. Finish the proof.

□

Problem 1. Consider a circle inscribed into an angle $\angle B'AC'$. Let P and Q be the points where the circle is tangent to the angle's sides. Let B and C be points on the rays AB' and AC' respectively so that the line through B and C is tangent to the circle at point T. Assume that A and the center of the circle O lie on different sides from BC. Let M be the point of intersection of CP and BQ. Show that A, T and M lie on the same line.

Problem 2. Let M be a point inside of $\triangle ABC$, and A_1, B_1, C_1 be points on its sides BC, AC and AB respectively so that $|CB_1| = \frac{1}{3}|CA|$, $CA_1| = 1/4|CB|$. Find the ratio in which C_1 divides AB.