The Geometry of Point Masses

October 20, 2008

Consider two point masses and their center of mass (the point of balance). Recall the *Archemes' principle of lever:*

The product of the mass and the distance to the center of mass is the same for both point masses:

$$m_1 \cdot d_1 = m_2 \cdot d_2.$$

Here are the properties of the center of mass of a system of points:

1. Any finite set of point masses has a center of mass. The center of mass is unique.
2. For two point masses, their center of mass lies on the segment joining the points and dividing the segment in the ratio which is inverse proportional to the corresponding masses.
3. The position of the center of mass of a system of point masses is not changed by replacing several point masses from the system with their total mass positioned at the center of mass of this subsystem.

Let mP denote the point mass m positioned at point P. Then:

1. $m_1 P_1 = m_2 P_2$ iff $m_1 = m_2$ and $P_1 = P_2$.
2. $m_1 P_1 + m_2 P_2 = (m_1 + m_2)P$, where P is the point on the segment $P_1 P_2$ such that $|P_1 P| \cdot m_1 = |P_2 P| \cdot m_2$. In other words, $|PP_1| : |PP_2| = m_2 : m_1$.

The point P above is called the *center of mass* of the point masses $m_1 P_1$ and $m_2 P_2$.

WARM-UP PROBLEMS:

1. Let Z be the center of mass of two point masses, $3P$ and $5Q$. Find m and the ratio $|PZ| : |ZQ|$.
2. Let Z be the center of mass of two point masses, $7A$ and mP. Find m and the ratio $|AZ| : |ZP|$ if $7A = mP = 10Z$.

Solve the following geometry problems using point masses:

1. Let AD be the median bisecting the side BC in $\triangle ABC$. Let $Z \in AD$ be a point on AD such that $|AZ| = |ZD|$. Find the ratio in which the line going through B and Z divides the side AC.

2. Let M be the point on the side AC of $\triangle ABC$ such that $|AM| = \frac{1}{3}|AC|$. Let N be the point on extension of the side BC beyond point B so that $|BN| = |NC|$. Let P be point of intersection of AB with MN. Find the ratios $|AP| : |PB|$ and $|NP| : |PM|$.

3. Let $ABCD$ be a convex quadrilateral. Let K, L, M, N be the midpoints of the sides AB, BC, CD and DA respectively. Show that the point O of intersection of KM and LN is the midpoint of both of these segments. Show that O is also the midpoint of the segment connecting the midpoints of diagonals of the quadrilateral.

4. In $\triangle ABC$, D is the midpoint of BC and E is divides AC in the ratio $1 : 3$. Let $K = BE \cap AD$. Find the ratios $|AK| : |KD|$ and $|BK| : |KE|$.

5. A line goes through the vertex A of triangle $\triangle ABC$ and the midpoint L of the median BB_1. In what proportion does this line divide the median CC_1? *(Hint:* This problem can not be solved in one step. First, select the masses so that L is center of mass. Using this, find the ration in which the extension of AL divides BC. Second, after you know the ratio, select new masses so that the point K of intersection of AL amd CC_1 is the center of mass, and finish the problem).

6. Let $ABCD$ be a parallelogram. Let l be the line going through D and crossing the segment AB in such a way that $|AK| = \frac{1}{n}|AB|$. In what ratio does this line divide the diagonal AC?

7. (Varignon's Theorem) If the midpoints of consecutive sides of a quadrilateral are connected, the resulting quadrilateral is a parallelogram.

8. Let $ABCD$ be a quadrilateral such that a circle can be insribed into it. Let $M \in AB$, $N \in BC$, $P \in CD$ and $Q \in AD$ be the points where the sides of the quadrilateral are tangent to the circle. Suppose that $|AM| = a$, $|BN| = b$, $|CP| = c$ and $|DQ| = d$. Let $Z = MP \cap NQ$. Find he ratios $|MZ| : |ZP|$ and $|QZ| : |ZN|$.

9. Let M and N be the points on the sides AC and BC of $\triangle ABC$ respectively, so that $|AM| : |MC| = 3 : 1$ and $|BP| : |PC| = 1 : 2$. Let $Q = AP \cap BM$. Given that area of $\triangle BPQ$ is equal to 1 square inch, find the are of $\triangle ABC$.

10. Let $PABC$ be the right triangular pyramid. (A pyramid is "right" its base is an equilateral triangle, and the altitude from the apex goes through the center of this triangle). Assume that a plane α intersects the pyramid in such a way that it divides the sides PA, PB and PC in the ratios $2 : 3$, $3 : 2$, and $4 : 1$ respectively. Find the ratio in which the plane divides the altitude PM of the pyramid.