INVARIANTS AND KNOTS

OCTOBER 22, 2008

1. Invariants

An *invariant* is a quantity that is left unchanged by a process. If you want to show that you cannot go from one situation to another using the respective process, one way to do it is to find an invariant that takes different values for the two situations.

Problem 1.1. Can you cover an 8×8 chessboard with 3×1 tiles?

Solution. No. There are 64 squares on the chessboard, which is not divisible by 3. Thinking in terms of invariants, consider the process of adding 3×1 tiles. The quantity

$$I = \text{ the remainder of the division: (number of squares)/3}$$

is an invariant, because it is unchanged by this process. Since we start out with $I = 0$, we cannot end up at $I = 1$. (The remainder of the division 100/3 is 1.)

Problem 1.2. From an 8×8 chessboard two diagonally opposite tiles are removed. Can the resulting board be covered using dominoes (2×1 tiles)? (Hint: Consider the invariant given by the number of black squares minus the number of white squares.)

Problem 1.3. Show that if every room in a house has an even number of doors, then the number of outside entrance doors must be even as well.

Problem 1.4. At first, a room is empty. Each minute, either one person enters or two people leave. After exactly 1000 minutes, could the room contain exactly 200 people?

Problem 1.5. On a tropical island there are 20 red chameleons, 13 blue chameleons and 10 green chameleons. When two chameleons of different color meet they change to the third color. Can all the chameleons eventually be of the same color? (Hint: Think about the number of red minus the number of blue chameleons.)

Problem 1.6. Can a 10×10 chessboard be covered with 4×1 tiles? (Hint: Write the numbers 3, -1, -1, -1, 3, -1, -1, -1, 3, -1 in the first row of the chessboard; then the same sequence of numbers in the second row, but shifted one spot to the left so that it starts with -1; then the same sequence in the third row, shifted two spots to the left, etc. This should help you find a suitable invariant.)

2. Knots

A *knot* is a closed piece of string in three-space. Knots are represented by *diagrams*: their projections to a plane, where at each crossing we specify which strand is the underpass and which one is the overpass.

Two knots are equivalent (that is, "the same") if they can be deformed into each other in three-space. For example, both diagrams in Figure 3 represent the unknot. Figure 1 shows a table of small knots.

Two diagrams represent the same knot if they can be related by a deformation process. Reidemeister proved that any deformation process consists of a sequence of moves, where each move is one of the three "Reidemeister moves" shown in Figure 2.

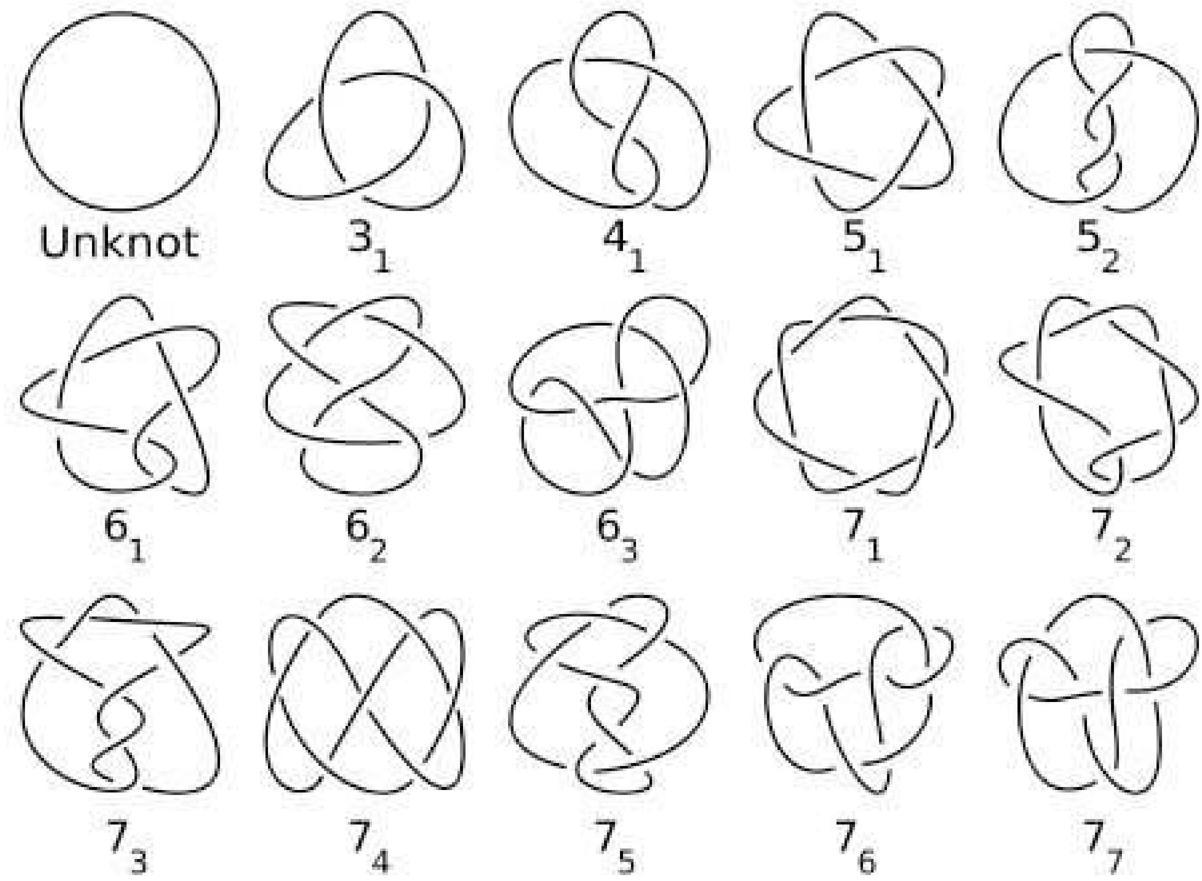

FIGURE 1. A table of small knots.

Problem 2.1. Explain which Reidemeister moves need to be used to go between the two diagrams in Figure 3.

We want to distinguish knots. For example, we seek to show that the trefoil (knot 3_1 in Figure 1) cannot be unknotted. The idea is to find an invariant of knots: a quantity associated to diagrams that is unchanged by Reidemeister moves. Then, if the trefoil and the unknot have different invariants, we would know that they cannot be deformed into each other.

Any diagram is made of several arcs (from one underpass to the next), drawn as uninterrupted curves in the respective diagram. For example, the first five diagrams in Figure 1 have $1, 3, 4, 5, 5$ arcs, respectively. There are three arcs meeting at each crossing: two from the underpass and one from the overpass.

We say that a diagram is 3-colorable if we can color its arcs in red, blue and yellow such that:

- All three colors are used;
- At each crossing, either one color or all three colors appear.

Problem 2.2. Which of the first five diagrams in Figure 1 are 3-colorable?

Given a diagram D, consider the quantity

$$I(D) = \begin{cases} 1 & \text{if } D \text{ is 3-colorable} \\ 0 & \text{otherwise.} \end{cases}$$

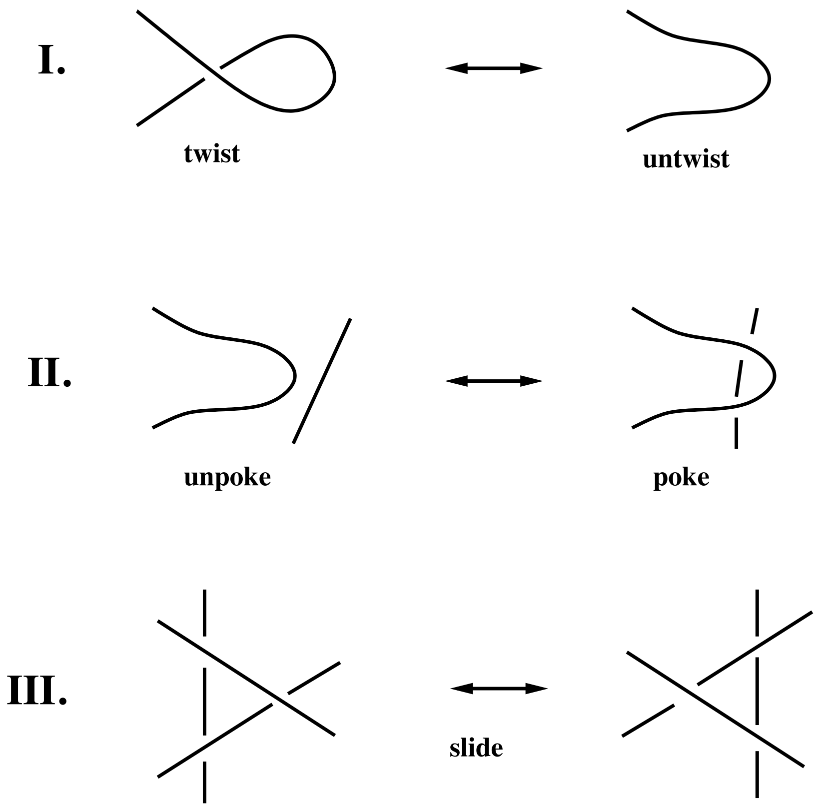

FIGURE 2. The three Reidemeister moves.

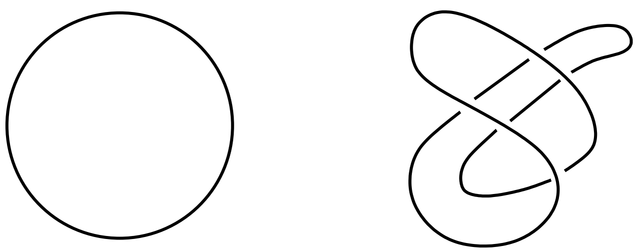

FIGURE 3. Two diagrams of the unknot.

Problem 2.3. Show that $I(D)$ does not change when we apply to a diagram one of the Reidemeister moves in Figure 2. Deduce that $I(D)$ is a knot invariant, and use this to conclude that the trefoil cannot be unknotted.

3. Links

A *link* is a union of knots in three-space. The knots can be linked with each other. Some examples of links are given in Figure 4. A link can have several components, but for simplicity we will only discuss links with two components.

Just as for knots, links are represented by their projections, and two diagrams represent the same link only if they can be related by a sequence of Reidemeister moves.

We seek to show that the three links in Figure 4 are all distinct. In order to do this, we develop a new invaraint called the *linking number*. We first choose an orientation on the link, that is, a direction on each of the two components. Then, each crossing in the link is either positive or

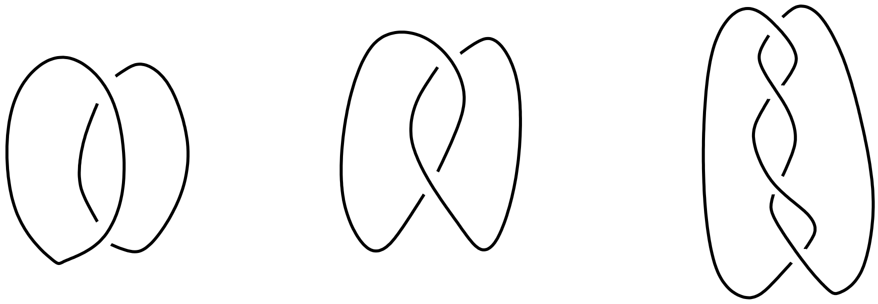

FIGURE 4. From left to right: the unlink, the Hopf link, and the T(2,4) link.

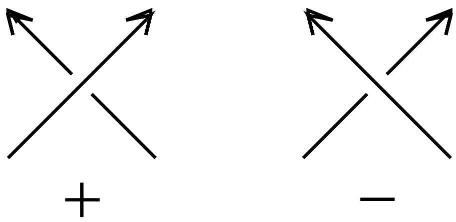

FIGURE 5. Positive and negative crossings.

negative, as shown in Figure 5. Therefore, to each crossing c we can associate a quantity $s(c) = +1$ or -1, according to the sign of the crossing. Let C be the set of crossings in the diagram where the two strands meeting at that crossing belong to different components. Set

$$lk(D) = \frac{1}{2}\sum_{c \in C} s(c),$$

that is, we're summing the quantities $s(c)$ over all crossings in C and divide by 2.

For example, in Figure 6, only the crossings marked with $+$ and $-$ are counted in C. The corresponding linking number is

$$\frac{1}{2}(3-1) = 1.$$

Problem 3.1. Choose orientations for the links pictured in Figure 4, and calculate the corresponding linking numbers.

Problem 3.2. What happens to the linking number when we change the orientation of one component? What if we switch the orientation of both components?

Problem 3.3. Show that $lk(D)$ does not change when we apply to a diagram one of the Reidemeister moves in Figure 2. Deduce that the absolute value $|lk(D)|$ is a link invariant, and use this to conclude that the the links pictured in Figure 4 are all different.

Problem 3.4. Calculate the linking number for the link pictured in Figure 7.

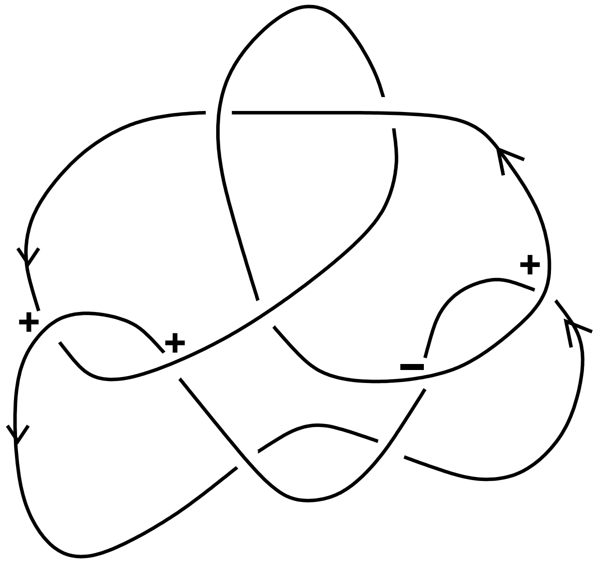

Figure 6. Computing the linking number.

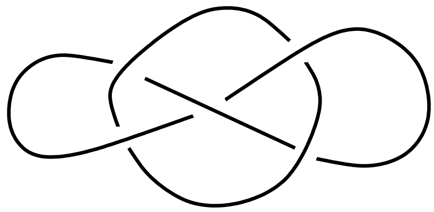

Figure 7. The Whitehead link.