Pigeonhole principle

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The following general principle was formulated by a famous German mathematician Dirichlet (1805-1859).

Pigeonhole principle:

Let there be n pigeonholes and k letters to be placed in the pigeonholes. If $k > n$, there is at least one pigeonhole containing at least two letters.

Here is another version:

If you have k rabbits sitting in n boxes, where $k > n$, then at least 2 rabbits are sitting in the same box.

1. Prove the Pigeonhole principle by contradiction.
2. In the movie "Cheaper by the Dozen", there are 12 children in the family.
   (a) Prove that at least two of the children were born on the same day of the week;
   (b) Prove that at least two family members (including mother and father) are born in the same month;
   (c) Assuming that there are 4 children's bedrooms in the house, show that there are at least 3 children sleeping in at least one of them.
3. The Pigeonhole Elementary school has 500 students. Show that at least two of them were born on the same day of the year.
4. There are $800,000$ pine trees in a forest. Each pine tree has no more than $600,000$ needles. Show that at least two trees have the same number of needles.

Generalized Pigeonhole principle:

If k rabbits are sitting in n boxes, where $k > n$, there is at least one box with more then $\frac{n}{k}$ rabbits.

Example: If you have 5 rabbits sitting in 2 boxes, then there must be 3 or more rabbits in at least one of the boxes.

5. Prove the generalized Pigeonhole principle.
6. There are 50 baskets of apples. Each basket contains no more then 24 apples. Show that there are at least 3 baskets containing the same number of apples.
7. Show that among any 4 numbers one can find 2 numbers so that their difference is divisible by 3. (Avoid considering the cases separately. Use Pigeonhole principle).
8. Show that among any $n+1$ numbers one can find 2 numbers so that their difference is divisible by n.
9. Show that for any natural number n there is a number composed of digits 5 and 0 only and divisible by n.
10. Given 12 different 2 digit numbers, show that one can choose two of them so that their difference is a two-digit number with identical first and second digit.
11. There are 10 segments marked on a segment of length 1. The total length of the marked segments is 1.1. Show that at least two of the marked segments have a common point.
12. There are 13 squares of side 1 positioned inside of the circle of radius 2. Show that at least 2 of the squares have a common point.