ANGLES AND CIRCLES

Preliminary information:

Inscribed angle: an angle such that its vertex lies on the circle and its sides intersect the circle.

Fact 0.1. *Let $\angle ABC$ be an insribed angle with vertex B being on the circle. Then*

- *if B and O are on the same side of AC, we have $\angle ABC = \frac{1}{2}\angle AOC$.*
- *if B and O are on different sides of AC, we have $\angle ABC = 180^0 - \frac{1}{2}\angle AOC$.*

Fact 0.2. *The angle between the chord AB and a tangent to the circle at point A is equal to half of the angle subtended by the arc AB.*

Fact 0.3. *Let A, B, C, D be 4 points in the circle, positioned in this order. Then*

(1) *The angle between the chords AC and BD equals to $\frac{\breve{AB}+\breve{CD}}{2}$.*
(2) *The angle between the chords AB and CD equals to $\frac{\breve{AD}-\breve{CD}}{2}$.*

Fact 0.4. *A quadrilateral $ABCD$ is inscribed iff either of the two conditions below holds:*

(1) $\angle ABC + \angle CDA = 180^0$.
(2) $AB + CD = BC + AD$.

Fact 0.5. *Let A be a point and l_1 and l_2 be two lines going through A and intersecting a given circle at B_1, C_1 and B_2, C_2. Show that $AB_1 \cdot AC_1 = AB_2 \cdot AC_2$. (Note that A can be either inside, or outside of the circle, or on the circle. In the latter case, if l_2 is tangent to the circle, i.e., $B_2 = C_2$, then $AB_1 \cdot AC_1 = AB_2^2$.*

(1) Consider the following two problems:
   (a) Let $\angle BAC$ be an angle whose vertex A lies outside of a circle with center O. Let M, N be the points of intersection of ray AB with the circle, and P, Q be the points of intersection of the ray AC with the circle. Prove that
   $$\angle BAC = \frac{\angle NOQ - \angle MOP}{2}.$$
   (b) Let $\angle BAC$ be an angle whose vertex A lies inside of circle with center O. Let M and N be the points of intersection of the ray AB and its extention beyond A with the circle. Let P and Q be the points of

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intersection of the ray AC and its extension beyond A with the circle. Prove that

$$\angle BAC = \frac{\angle NOQ + \angle MOP}{2}.$$

(c) Think how the two previous parts of the problem are similar. What is happening when A is *on* the circle?

(2) A point P is inside of an acute angle $\angle BAC$. Let $C_1 \in AB$ and $B_1 \in AC$ be such that $PB_1 \perp AC$ and $PC_1 \perp AB$. Show that

$$\angle C_1AP = \angle C_1B_1P.$$

(3) The centers I of inscribed and O of circumscribed circles of triangle $\triangle ABC$ are symmetric to each other with respect to the side AB. Find the angles of $\triangle ABC$.

(4) Let $ABCD$ be a quadrilateral inscribed in a circle with center O. Let M be the middle of the arc $\breve{AB}$. Let $E = MC \cap AB$ and $K = MD \cap AB$. Show that the quadrilateral $KECD$ is inscribed.

(5) Let $\triangle ABC$ be a triangle, O be the center of its circumscribed circle, and AH be an altitude. Show that $\angle BAH = \angle OAC$.

(6) Let $ABCD$ and $A'B'C'D'$ be equilateral trapezoids inscribed in the same circle. Given that the respective sides of the trapezoids are parallel to each other, show that $AC = A'C'$.

(7) Let AB and CD be two diameters of a circle with center O. Let M be a point on the circle. Let MP and MQ be the perpendiculars dropped from M to AB and CD respectively. Show that the length of PQ does not depend on the position of M.

(8) In $\triangle ABC$ the sides AC and BC are not equal to each other. Show that the angular bisector of the angle C bisects the angle between the median and the altitude from the same point if and only if $\angle C = 90^0$.

(9) In triangle $\triangle ABC$ the median, the bisector and the altitude starting at C divide the angle $\angle C$ into 4 congruent angles. Find the angles of this triangle.

(10) Consider two circles which intersect in such a way that AB is their common chord. Let $P \in AB$ be a point on this chord. Let KM and LN be the chords of the first and the second circles respectively, so that $KM \cap LN = P$. Show that the quadrilateral $KLMN$ is inscribed.