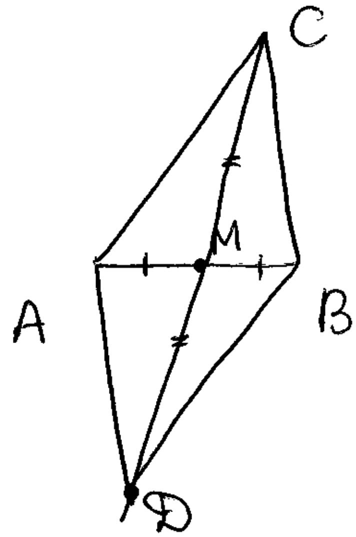
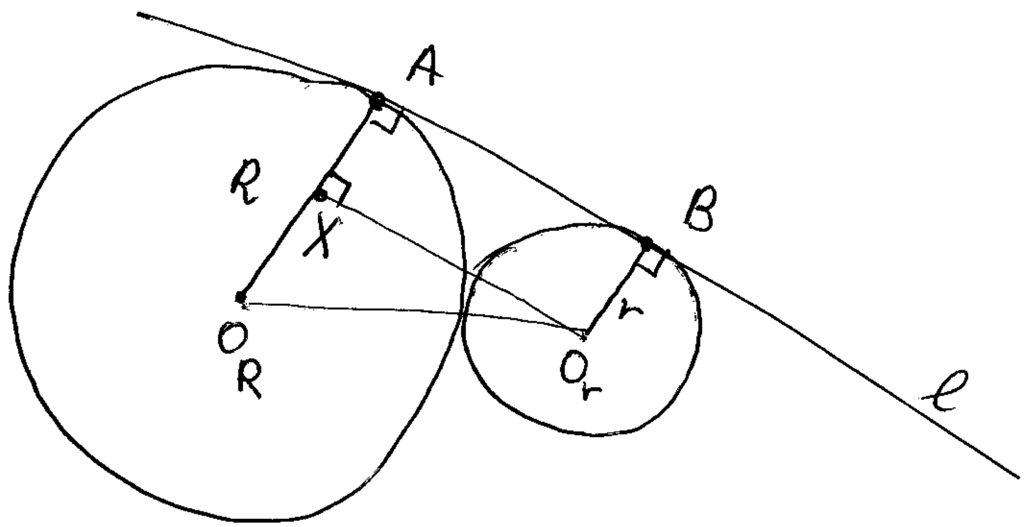
Problem 1



Construct D as follows: continue CM past M and make $DM = CM$. Then $\angle CMA = \angle BMD$, $BM = MA$, $CM = MD$, so $\triangle DMB = \triangle CAM$. Thus $BD = CA$. Consider now $\triangle CDM$.

In this triangele, $CD \leq DB + CB$ (by triangle inequality). Thus $CM = \frac{1}{2}(CD) \leq \frac{1}{2}(CB + DB)$.
But $BD = CA$ so $CM \leq \frac{1}{2}(CB + CA)$, as claimed.

Problem 2



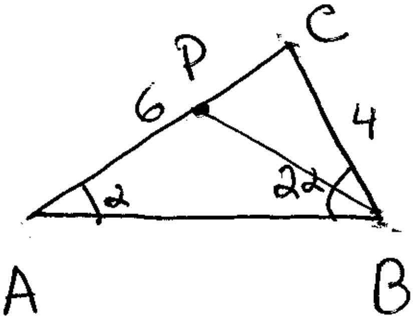
Without loss of generality, assume that $R > r$. Also, denote by O_R and O_r the centers of the corresponding circles. Since the line ℓ is tangent to the two circles, $O_RA \perp \ell$ and $O_rB \perp \ell$. Draw a perpendicular from O_r onto O_RA; denote it by O_rX. Since $AB \perp O_RA$ and $O_rX \perp O_RA$, ABO_rX is a rectangle and so $O_rX = AB$. Now, consider the right triangle O_RXO_r. In it, $O_RX = R - r$ and $O_RO_r = R + r$. Thus

$$AB = O_rX = \sqrt{(R+r)^2 - (R-r)^2}$$ by Pythagoras' Theorem

So $$AB = \sqrt{R^2 + 2Rr + r^2 - (R^2 - 2rR + r^2)} =$$

$$= \sqrt{2Rr + 2Rr} = 2\sqrt{Rr}.$$

Problem 3



Consider the bisector of the angle $\angle CBA$. Call it BP. Then $\triangle BCP$ is similar to $\triangle ACB$.

Indeed, $\angle BCP = \angle ACB$ and $\angle PBC = \angle BAC$, so all angles of $\triangle BCP$ are equal to corresponding angles in $\triangle ACB$. Thus $PB : AB = CB : AC =$ $= PC : BC$. Let us denote PB by y. Then $PC = AC - AP = 6 - y$ (since $AP = PB$, because $\angle CAB = \angle PBA$). Hence

$$CB : AC = PC : BC, \text{ so}$$

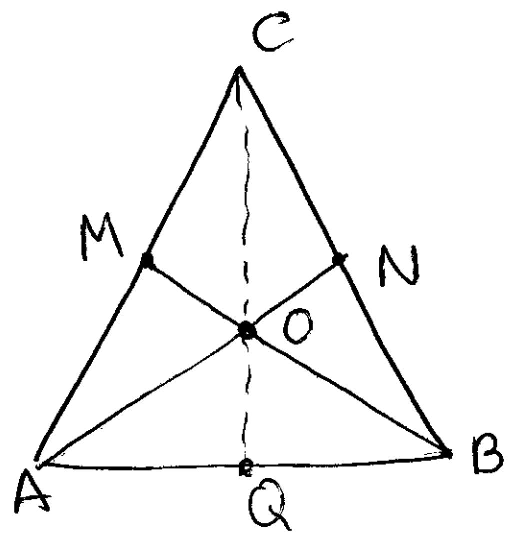
$$4 : 6 = (6-y) : 4$$

$$\Rightarrow \quad 6 - y = 8/3, \quad y = 10/3.$$

Now, $PB : AB = y : AB = 4 : 6$, so

$$AB = 3/2 \cdot 10/3 = 5.$$

Problem 4

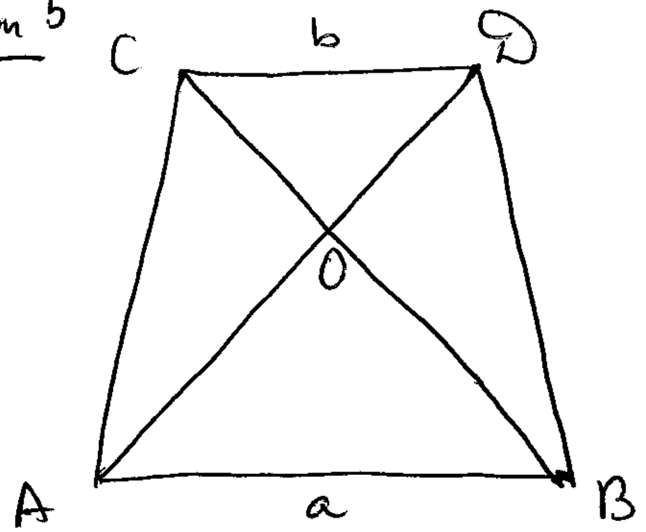


Let us assume that AN and BM are medians and AN = BM. Let O = point of intersection of AN and BM. Let CQ = third median.

Since the medians of a triangle intersect in a single point, CQ passes through O. Thus OQ is a median of ΔAOB. Since the point of intersection of medians divides them in the ratio 1:2, $OB = 2/3\ MB = 2/3 AN = AO$, so ΔAOB is isosceles, $AO = OB$. But then its median OQ is $\perp$ AB. But then $CQ \perp AB$. Thus ABC is also isosceles.

(Once we know $\angle OQB = 90°$, $\angle OQB = \angle OQA$ and $\Delta\ CCQ = BCQ$ since $AQ = QB$, $CQ = QC$ and $\angle OQB = \angle OQA$)

Problem 5



Let O = intersection of diagonals of the trapezoid.

Assume that $AB = a$, $CD = b$. Then because $AD \perp BC$, and because the trapezoid is isosceles, $AO = OB$ and $CO = OD$ (indeed, $\angle ABD = \angle BCD$, so $\angle OAB = \angle OBA$). Thus $AO = a/\sqrt{2}$, $OB = a/\sqrt{2}$, $CO = b/\sqrt{2}$ and $DO = b/\sqrt{2}$ by Pythagoras' theorem.

From this we compute:

$$\text{area}(\triangle AOB) = \tfrac{1}{2} a/\sqrt{2} \cdot a/\sqrt{2} = a^2/4$$

$$\text{ar}(\triangle COD) = \tfrac{1}{2} b/\sqrt{2} \cdot b/\sqrt{2} = b^2/4$$

$$\text{ar}(\triangle BOD) = \tfrac{1}{2} a/\sqrt{2} \cdot b/\sqrt{2} = ab/4$$

$$\text{ar}(\triangle ACO) = \text{ar}(\triangle BOD) = ab/4.$$

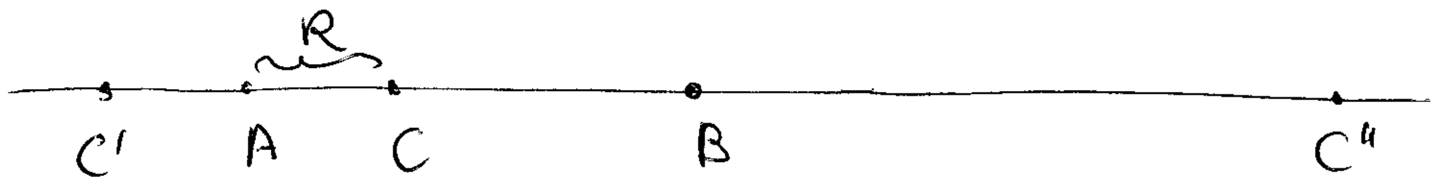
So

$$\text{area}(ABCD) = a^2/4 + 2ab/4 + b^2/4$$

$$= (a/2 + b/2)^2 = m^2 \text{ where } m = \frac{a+b}{2}.$$

But m = length of midline, so area = (length of midline)^2.

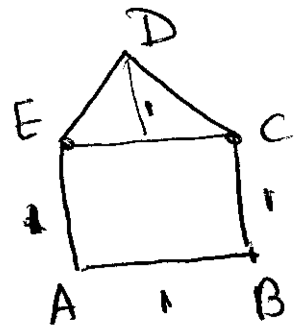
Problem 6 The statement of the problem is false. For example, consider the 180° rotation about A followed by 180° rotat[ion] about B. Call this transformation T.



Then clearly the line AB is sent to itself. Since the center of rotation is the unique point fixed by the rotation (unless it's rotat[ion] by 0°), the center must be on AB (clearly T moves A, so T cannot be rotat[ion] by 0°). If C is distance R to A, rot. by 180° about A maps it to C', and rot. by 180° maps C' to C''. So $T(C) = C''$, where $|BC''| = |BC'| = |AB| + R$. Thus $T(C)$ = pt on AB which is to the same side as C, and which is dist $|AB| + |AC|$ to A. Thus all pts on AB are translated by $|AB|$!

~~So no points are fixed~~ So T cannot ~~have~~ be a rotation, since no rotation can take a line to itself, shifting all of its points.

Problem 7) ctd This leads to the possibilities:



$(x,y) = (A,E), (A,D), (A,C), (A,B)$
$(E,D), (E,C), (E,B),$
$(D,C), (D,B), (C,B).$

The maximum is $|AD| = |BD| = \sqrt{5}$.

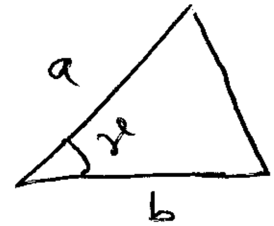
Claim If x, y are any two points in Q_i, $i=1,2$, then $\text{dist}(x,y) \leq \sqrt{5}$.

Indeed, we can embed Q_i into R_i.

Now assume that there are 6 pt chosen in the rectangle. Then two must lie in the same region R_1, R_2, R_3, Q_1 or Q_2 (by the pigeon-hole principle). But then the distance between these points must be $\leq \sqrt{5}$.

Problem 8 Let α, β, γ be the three angles.

Clearly one of α, β, γ must be $\leq 60°$ (since $\alpha+\beta+\gamma = 180°$). The area of the triangle is given by $A = \frac{1}{2} ab \sin \gamma$



We may assume (by relabeling angles, if necessary), that γ is the smallest angle, so $\gamma \leq 60°$. Then

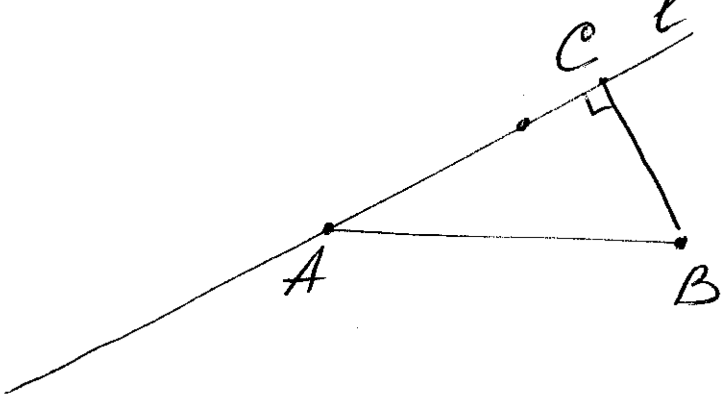
$$A \leq \frac{1}{2} ab \sin \gamma \leq \frac{1}{2} a \cdot b \cdot \sqrt{3}/2.$$

But $a, b \leq 1$ so $A \leq \frac{1}{2} \sqrt{3}/2 = \sqrt{3}/4$.

Note: It's now clear that A is maximum (subject to the assumption that all sides ≤ 1) when the smallest angle is maximal, i.e., all angles are $60°$, i.e. A is an equilateral triangle.

Problem 9.

Fig. 1:

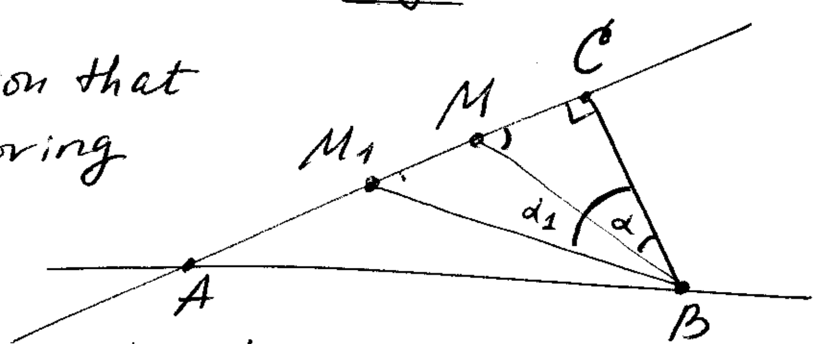


Let l be a line through A.
Let $C \in l$ be such a point that $BC \perp l$.

<u>Claim.</u> A point $M \in l$ is in the desired set of points iff M lies on the segment AC.

<u>Proof of Claim.</u>

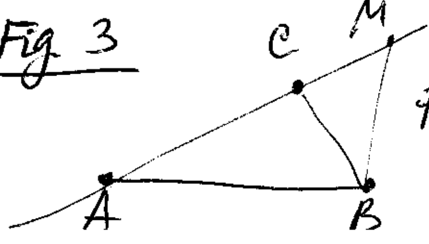
Fig. 2:



Let $M \in AC$.
Let M_1 is a position that you reach while moving from M to A along l.
Since $|BM_1| = \dfrac{|BC|}{\cos \alpha_1} > \dfrac{|BC|}{\cos \alpha} = |BM|$,
it follows that the distance from you to B always increases. (Here $\angle CBM = \alpha < \alpha_1 = \angle CBM_1$)

Let $M \in l$, M be further from A then C (with $C \in AM$)

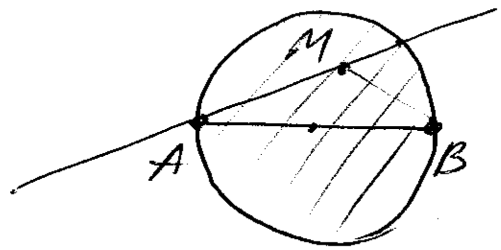
Fig 3



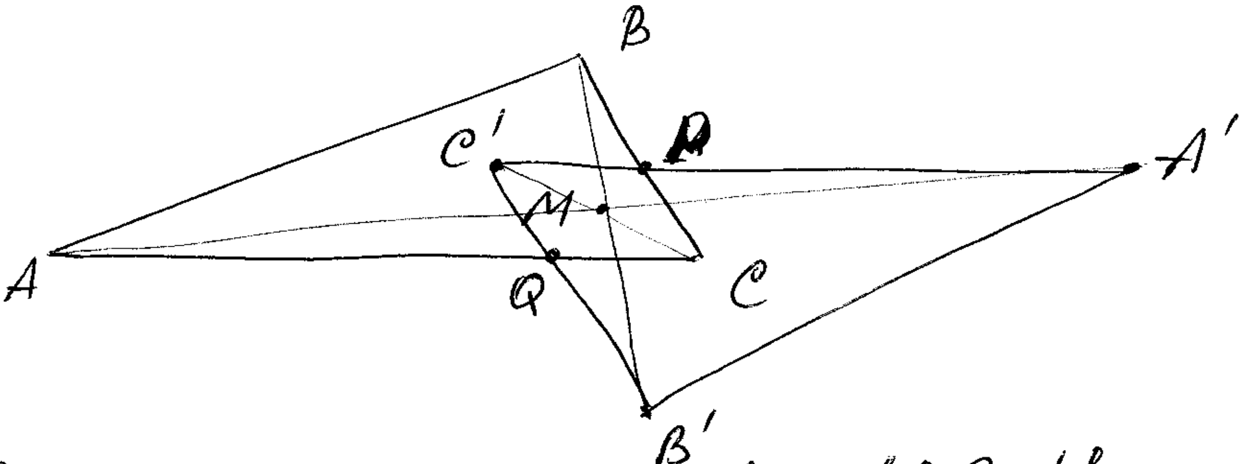
Since $|BC|$ is the shortest distance from B to a pt on l, M can not belong to the desired set. It is also easy to see that

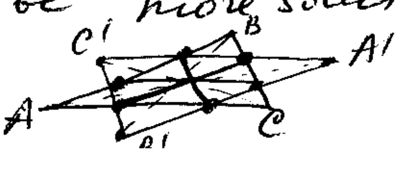
Points "below" the line AB lying on l can not be in the set.

By varying the line l, we get that the set of all pts M consists of such M that $\angle BMA \geq 90°$. This is a disc with AB serving as a diameter.



Problem 10.



Reflect all the vertices of $\triangle ABC$ through the point M. We get a new $\triangle A'B'C'$.
Let $P = BC \cap A'C'$, $Q = AC \cap B'C'$.
By construction, $\triangle ABC$ & $\triangle A'B'C'$ are symmetric to each other. Therefore P & P' are symmetric to each other w.r.to M. Hence, $|PM| = |MQ|$, and P, M, Q lie on the same line.
(Note that depending on position of M there could be more solutions. Such is the case, e.g., for 

). Prove as an exercise that there is always at least ...