Problem Solving Welcome: Solutions
(LAMC, Fall 2008)

September 21, 2008

1. A team of bakers consists of an experienced chef and 9 student bakers. During the day each student baker decorated 15 cakes. The number of cakes decorated by the chef is 9 more then the team's average.
   How many cakes were decorated by the whole team on this day?

   SOLUTION: Consider the chef's output. If we take the 9 "extra" cakes he bakes compared to the team's average and distribute it among the student bakers (one to each student), then every student has 16 cakes, and the chef bakes remains with the average number of cakes. These two quantities should be equal. Thus, the average number of cakes is 16 and the total number of cakes baked is $16 \cdot 10 = 160$. The number of cakes baked by the chef is $16 + 9 = 25$. The total number of cakes can also be obtained as
   $$15 \cdot 9 + 25 = 135 + 25 = 160$$

   One can, of course, solve this problem using algebra. For example, denote the average number of cakes by x. Then
   $$15 \cdot 9 + (x + 9) = 10x.$$
   Solving for x, we get $x = 16$ and then proceed as in the first solution. However, the first solution is much more beautiful. The idea to solve the problem in this way might come from the fact that the number of students (9) equals to the number of extra cakes baked by the chef compared to the team's average. If the numbers would not have this property, the first solution would not work as smoothly.

2. Find all x such that $x^{x^3} = 3$. (Here x is raised to the power x^3).

   SOLUTION: Clearly, a standard method is not going to work here. First, note that x has to be positive. Notice that $x = \sqrt[3]{3}$ does satisfy the equation:
   $$(\sqrt[3]{3})^{(\sqrt[3]{3})^3} = 3.$$

We are not done yet, as we had to find *all* solutions. In this problem, I claim that there are no other solutions. Indeed, if $x > \sqrt[3]{3}$, then $x^3 > 3$ and, therefore,

$$x^{x^3} > (\sqrt[3]{3})^{x^3} > (\sqrt[3]{3})^3 = 3.$$

Similarly, for $x < \sqrt[3]{3}$ all the opposite inequalities are true.
Only now we can say that we found all the solutions, $x = \sqrt[3]{3}$ to the equation.

3. At noon both the hour and the minute hands of an analog clock are pointing up, and thus coincide. When will the two hands be pointing in the same direction (coincide) next time?
SOLUTION: The hour and the minute hands move in such a way that when the minute handle goes around the circle, the hour handle covers 1 hour. Thus, the minute handle moves 12 times as fast as the hour handle. The next moment that the two hands coincide is the time x minutes after 1 pm, where x is such that

$$x = 5 + \frac{x}{12},$$

i.e., $\frac{11}{12}x = 5$. (This condition simply means that when the minute handle covers x minutes, the hour hand covers $\frac{x}{12}$ "minutes" starting at "5 minute mark", and the hands coincide at that moment). Therefore, $x = 5\frac{5}{11}$ minutes. Thus, the time when the hands coincide next time is $5\frac{5}{11}$ minutes after 1 p.m.

4. Without using a calculator, decide which number is bigger: ${}^2\sqrt{2}$ or ${}^5\sqrt{5}$?
SOLUTION: Raise both numbers to the degree $2 \cdot 5 = 10$:

$$(\sqrt[2]{2})^{10} = 2^5 = 32 \quad > \quad 25 = 5^2 = (\sqrt[5]{5})^{10}.$$

Therefore, the first number is bigger.

5. My favorite 3 digit number is such that if you subtract 7 from it, the result is divisible by 7; if you subtract 8 from it, the result is divisible by 8; if you subtract 9 from it, the result is divisible by 9. What is my favorite 3 digit number?
SOLUTION: Let a be the number we are looking for. Since $a-7$ is divisible by 7, then so is a. Similarly, a is divisible by 8 and by 9. Therefore a is divisible by $7 \cdot 8 \cdot 9 = 56 \cdot 9 = 504$. Since the only 3-digit number divisible by 504 is 504, this must be the answer to the problem.

6. What is the remainder that you get when the number $13^{16} - 2^{25} \cdot 5^{15}$ is divided by 3? Can you find the remainer of division of the same number by 37?
SOLUTION: First, the following theorem is useful:

The sum (product) of any two natural numbers has the same remainder,

when divided by 3, as the sum (product) of their remainders.
To see why this is true, denote the two numbers by N_1 and N_2 and write them as $N_1 = 3k_1 + r_1$ and $N_2 = 3k_2 + r_2$ respectively. Compute the sum and the product and show that the statement is true.
Now let's use this to solve the first part of the problem.
When 13 is divided by 3, the remainder is 1. Therefore, the remainder of 13^{16} when divided by 3 is the same as that of 1^{16}, i.e., 1.
When 2 is divided by 3, the remainder is 2. When 5 is divided by 3 the remainder is also 2. Thus, the remainder of $2^{25} \cdot 5^{15}$ when divided by 3 is the same as that of $2^{25+15} = 2^{40}$.
Claim: even powers of 2 have remainder 1 when divided by 3, and that odd powers of 2 have remainder of 2 when divided by 3.

Therefore, 2^{40} has remainder of 1 when divided by 3.
Since the remainders of 13^{16} and $2^{25} \cdot 5^{15}$ are the same when divided by 3, the remainder of their difference is 0. In other words, the number is divisible by 3.

7. A farmer sold half of his apples and half of an apple to the first customer; after that he sold half of the remaining apples plus another half of an apple to the second customer; etc. After he sold half of the remaining apples and half of an apple to the 7th customer, there were no more apples left. How many apples did the farm have in the beginning?
SOLUTION: Let x be the number of apples the farmer had in the beginning. Compute the numbers of apples sold at each sale:
First sale: $\frac{x+1}{2}$; Second sale: $\frac{\left(x-\frac{x+1}{2}\right)+1}{2} = \frac{x+1}{4}$. Similarly, third sale is the sale of $\frac{x+1}{8}$ apples, etc. The $n$the sale is the sale of $\frac{x+1}{2^n}$ apples. The last sale is the sale of $\frac{x+1}{128}$ apples. Since after the last sale no more apples were left, we get the equation:

$$x\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}\right)+\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}\right)=x.$$

Solving for x, we get $x = 127 = 2^7 - 1$ apples.

8. Consider the sequence of numbers:

$$1,\ 3,\ 7,\ 15,\ 31,\ \ldots$$

   (a) Determine how each next number is obtained from the previous one.
   (b) Find the 10th term.
   (c) Find the 100th term.

   SOLUTION: It is easy to see that each next term is equal to twice the previous term plus one. Thus, if a_n denotes nth term, then

$$a_n = 2a_{n-1} + 1$$

You can also notice that the numbers in the sequence are by 1 less then the powers of 2. More precisely,

$$a_n = 2^n - 1.$$

One can prove this by induction[1] as follows:

- *(Base case)* First, for $n = 1$ the formula is true: $a_1 = 2^1 - 1 = 1$.
- *(Inductive Step)* Assume the the formula is true for $n-1$, i.e., $a_{n-1} = 2^{n-1} - 1$. We need to check that this implies that the formula is also true for a_n. Since

$$a_n = 2(a_{n-1}) + 1,$$

we get

$$a_n = 2(2^{n-1} - 1) + 1 = 2^n - 1.$$

Thus, the nth term is $2^n - 1$. In particular, the tenth term is $2^{10} - 1 = 1023$ and the 100th term is $2^{100} - 1$.
(Did you notice the similirity between the sequence in this and in the previous problem?)

9. The product of two positive numbers is equal to 100.

   (a) How small can the sum of these numbers be?

   (b) How large can the sum of these numbers be?

   (c) What are the answers to the questions above if both numbers are required to be integers (whole numbers)?

SOLUTION: Solution to this problem is based on the inequality between the *arithmetic mean* $\frac{a+b}{2}$ of two numbers a and b , and their *geometric mean* $\sqrt{a \cdot b}$*:*

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

In words, the geometric mean is no bigger then the arithmetic mean. Moreover, the equality is reached if and only if $a = b$.
This inequality implies that

$$2\sqrt{ab} \leq a + b.$$

This implies that if $a \cdot b = 100$, then the smallest the sum can be is $2\sqrt{100} = 20$. This is achieved when $a = b = 10$.
On the other hand, the sum can be as large as you want. For example, suppose someone asks you if you can make the sum bigger then a (huge) positive number M (a million, or 10^{10}, or anything else). Take

[1] If you are not familiar with proofs by induction just yet, don't worry, we will definitely learn about them this year

$a = M$ and $b = \frac{100}{M}$. Then the product $ab = 100$ and the sum $a + b = M + \frac{100}{M} > M$.
If you require a and b to be integers, the minimal sum is the same (for $a = b = 10$, $a + b = 20$). The maximal sum is reached when one of the number is as large as possible. If $a = 100$ and $b = 1$, the sum is the maximal possible, $a + b = 101$.

10. Compute without using a calculator:

$$20^2 + 19^2 + \ldots + 11^2 - 10^2 - 9^2 - \ldots - 1^2$$

Solution: Regroup the terms in the following way:

$$(20^2 - 10^2) + (19^2 - 9^2) + \ldots + (11^2 - 1^2)$$

Using the formula $a^2 - b^2 = (a - b) \cdot (a + b)$ for each of the expressions in the brackets and factoring out 10, we get

$$10 \cdot (30 + 28 + \ldots + 14 + 12).$$

Computing the sum of the (finite) arithmetic progression, we get the final answer, 2100.