Instructions. You may consult any source you wish (books, friends, the Internet, ...) You should work on these problems in advance at home and present your solutions to one of the docents during the math circle meeting. However, in your solutions, you may only rely on basic algebra (and facts like $x^2 \geq 0$ for all x) or on problems that you have already solved. You may not, for example, rely on a theorem you found in a book (unless you prove it in the course of your solution).

Quadratic polynomials.

An expression of the form $ax^2 + bx + c$ with $a \neq 0$ is called a quadratic polynomial in x.

Problem 1. Show that the equation $ax^2 + bx + c = 0$ has solutions if and only if the discriminant $D = b^2 - 4ac$ satisfies $D \geq 0$. Show that in this case the solutions are given by the formula $x_{1,2} = \frac{1}{2a}(-b \pm \sqrt{D})$.

Problem 2. Show that if x_1 and x_2 are the two solutions to $ax^2+bx+c = 0$, then $x_1x_2 = c/a$ and $x_1 + x_2 = -b/a$. Also show that the distance from x_1 to x_2 is given by $\sqrt{D}/a$.

Problem 3. Let $a > 0$, and let $y = ax^2 + bx + c$. Show that there exists some $R > 0$ so that $y > 0$ whenever $|x| > R$.

Problem 4. Solve the following equations: (a) $t^4 + t^2 - 1 = 0$; (b) $t - 2/t = 0$.

Problem 5. Let $y = ax^2 + bx + c$, $\hat{y} = \hat{a}x^2 + \hat{b}x + \hat{c}$. Assume that $a > \hat{a} > 0$. Show that there exists some R so that $y > \hat{y}$ whenever $|x| > R$.

Problem 6. A function $f(x)$ is called (strictly) *convex* if

$$f(\alpha s + (1-\alpha)t) < \alpha f(s) + (1-\alpha)f(t) \quad \text{for any } s, t \text{ and } 0 < \alpha < 1.$$

(a) Explain why the following is true for the graph of a convex function f:
if (x_1, y_1) and (x_2, y_2) belong to the graph, then the straight line segment joining these two points lies above the graph of f.
(b) Show that $f(x) = ax^2 + bx + c$ is convex if $a > 0$.

Problem 7. If $f(x)$ is a function, we say that x is a *minimum* of f if $f(\hat{x}) \geq f(x)$ for any $\hat{x}$.
(a) Give an example (a graph is sufficient) of a function having exactly two minima.
(b) Show that if f is convex, it can have at most one minimum.
(c) Give examples of a convex function (a graph would suffice) having no minima and of a convex fuction (a graph would suffice) having exactly one minimum.
(d) Show that if $f(x) = ax^2 + bx + c$ and $a > 0$, then $f(x)$ has a unique minimum, at $-b/2a$.

Problem 8. Assume that $y = ax^2 + bx + c$, $a > 0$, and that for some x, the value of y is strictly less than zero. Show that then $ax^2 + bx + c$ has exactly two roots.

Problem 9. For which values of C does the system of equations

$$\begin{cases} x^2 + y^2 + z^2 = C \\ xy + yz + xz = C \\ x^2y^2z^2 = C \end{cases}$$

have a unique solution? Find this solution.

Problem 10. Let p_1, p_2, p_3 be three distinct points on the plane. Can one always draw a parabola that passes through these three points? If not, what is a reasonable assumption on p_1, p_2, p_3 that guarantees that you can do this?

Problem 11. Find all prime numbers p and q such that the equation

$$x^2 - px - q = 0$$

has a solution which is a prime number.

Problem 12. Without solving the equation $ax^2 + bx + c = 0$, where $a \neq 0$ and $D > 0$, find the sum of the squares of its roots.

Problem 13. (New problem. Please, note the change)

Consider the graph of the function $y = ax^2$, where $a > 0$. Show that there is a point $F = (0, f)$ (called *focus* of the parabola) and a line $y = -l$ (called the *directrix* of the parabola) such that for any point $(x,\ ax^2)$ on the parabola the distances from this point to the focus and to the directrix are equal to each other.

Do you think such a special point and a special line would exist for any other parabola? Why?

Problem 14. Write down a quadratic equation with integer coefficients such that its roots are equal to $\frac{1}{2}$ and $\frac{3}{7}$.

Problem 15. Let x_1 and x_2 be the roots of the quadratice equation

$$x^2 - 13x - 17 = 0.$$

Write down a quadratic equation whose roots are $2 - x_1$ and $2 - x_2$. Please, solve this problem without finding x_1 and x_2 explicitely.

Problem 16. Solve the following equation

$$\frac{x^2 + 2x + 7}{x^2 + 2x + 3} = 4 + 2x + x^2$$

Problem 17. (Please, note that the problem has been changed. The earlier problem does not have a solution as stated).

The product of the digits of a 2-digit number is equal to twice the sum of the digits plus 6. When the number is divided by the sum of its digits, the quotient is equal to 4 and the remainder is 3. Find the number.

Problem 18. Solve the equation

$$(a - 1)x^2 + 2(a + 1)x + (a - 2) = 0$$

for all values of a.