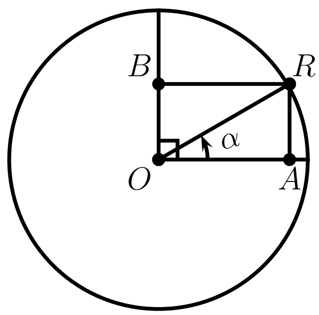
Instructions. You may consult any source you wish (books, friends, the Internet, ...) You should work on these problems in advance at home and present your solutions to one of the docents during the math circle meeting. However, in your solutions, you may only rely on statements called "Fact", "Definition" below, or on problems that you have already solved. You may not, for example, rely on a theorem you found in a book (unless you prove it in the course of your solution).

1. Trigonometry.

Definition 1. Let α be a number. Then $\cos\alpha$ is defined to be the signed length of OA and $\sin\alpha$ is defined to be the signed length of OB in the figure below. The radius of the circle is $|OR| = 1$ and $OA \perp OB$. The angle α is measured in radians, i.e., the right angle corresponds to $\alpha = \pi/2$.



Problem 1. Prove that $\sin^2\alpha + \cos^2\alpha = 1$.

Problem 2. Prove the following identities:

$$\begin{aligned} \sin(\alpha+\pi/2) &= \cos\alpha; \qquad \cos(\alpha+n\pi) = (-1)^n\cos\alpha, \qquad n = 0, \pm1, \pm2, \dots; \\ \cos(\alpha+\pi/2) &= -\sin\alpha; \qquad \cos(-\alpha) = \cos\alpha; \qquad \sin(-\alpha) = -\sin\alpha. \end{aligned}$$

Problem 3. First, prove the following identity (using any geometric method)

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Then use it to show that

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha,$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta, \qquad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Problem 4. Prove: $\sin 2\alpha = 2\sin\alpha\cos\alpha$, $\qquad \cos 2\alpha = 2\cos^2\alpha - 1$.

Problem 5. Show that $\sin 75^0 = \frac{\sqrt{6}-\sqrt{2}}{4}$ and $\cos 75^0 = \frac{\sqrt{6}+\sqrt{2}}{4}$.

Problem 6. Find expressions in radicals (similar to the ones for 75^0 above) for $\sin 15^0$ and $\cos 15^0$.

Problem 7. Show that $\sin^2\dfrac{\alpha}{2} = \dfrac{1-\cos\alpha}{2}$, $\qquad \cos^2\dfrac{\alpha}{2} = \dfrac{1+\cos\alpha}{2}$.

Problem 8. Show that:

$$\begin{aligned} 2\cos\alpha\cos\beta &= \cos(\alpha+\beta) + \cos(\alpha-\beta), \qquad 2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta), \\ 2\cos\alpha\sin\beta &= \sin(\alpha-\beta) + \sin(\alpha-\beta). \end{aligned}$$

Problem 9. Prove the identity

$$\sin^2\alpha - \sin^2\beta \;=\; \sin(\alpha+\beta)\cdot\sin(\alpha-\beta).$$

Can you find a similar identity for the difference of squares of cosines?

Problem 10. Compute the numerical value of the product $\cos 20^0 \cos 40^0 \cos 80^0$ without using a calculator.

Problem 11. Solve the equations: (a) $\sin 6x + \sin 4x = 0$, (b) $\sin x - \cos 2x = 0$.

Problem 12. Solve the equations: (a) $\sin x + \cos x = 1 - \sin 2x$, (b) $\sin^4 x + \cos^4 x = \frac{7}{2}\sin x \cos x$.

Problem 13. Solve the following system of equations:

$$\begin{cases} \sin^2 x + \sin^2 y = 1/2 \\ x - y = 4\pi/3 \end{cases}$$

Definition 2. The tangent and cotangent functions are given by

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}, \qquad \cot\alpha = \frac{\cos\alpha}{\sin\alpha}$$

whenever these expressions make sense (i.e., we are not dividing by zero).

Problem 14. (Uniformization) Show that

$$\begin{aligned} \sin\alpha &= \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \\ \cos\alpha &= \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \\ \tan\alpha &= \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}} \end{aligned}$$

These formulas allow us to represent all trigonometric functions as rational expressions of a single function $\tan\frac{\alpha}{2}$.

Problem 15. (Pythagorean triples via uniformization). Let (a,b,c) be a *Pythagorean triple*, i.e., a triple of positive integers such that $a^2+b^2=c^2$. Then there is a right triangle with legs a/c and b/c, and hypotenuse 1. So each acute angle of this triangle has a rational sine, cosine and tangent.

We can find Pythagorean triples by starting with $\tan\frac{\alpha}{2}$ being some rational number, and then determining $\sin\alpha$ and $\cos\alpha$ (which will also be rational).

Let $\tan\frac{\alpha}{2} = \frac{p}{q}$. Find the corresponding Pythagorean triple.

Find three simple examples of Pythagorean triples using this method.

Problem 16. Show that $\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}, \qquad \tan(\alpha-\beta) = \dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}.$

Problem 17. Show that $\tan(\alpha+\beta+\gamma) = \dfrac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-(\tan\alpha\tan\beta+\tan\beta\tan\gamma+\tan\alpha\tan\gamma)}$.

Problem 18. Show that if $\alpha+\beta+\gamma=\pi$, then $\tan(\alpha)+\tan(\beta)+\tan(\gamma)=\tan(\alpha)\tan(\beta)\tan(\gamma)$.

Problem 19. (Elementary Symmetric Polynomials)

A polynomial $P(x_1,\ldots,x_n)$ in variables $x_1,\ldots,x_n$ is called *symmetric* if it is not changed when any two of the variables are permuted, i.e.

$$P(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_n)=P(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_n)$$

for all x_i, x_j and all i,j. For example, $P(x,y,z)=xyz(3xy+3xz+3yz)$ is a symmetric polynomial in 3 variables.

It turnes out that any symmetric polynomial is "built" out of *elementary symmetric polynomials* $E_k(x_1,\ldots,x_n)$, which are the simplest possible symmetric polynomials. For example, for $n=2$:

$$\begin{aligned} E_1(x_1,x_2) &= x_1+x_2 \\ E_2(x_1,x_2) &= x_1x_2 \end{aligned}$$

For $n=3$ the elementary symmetric polynomials are

$$\begin{aligned} E_1(x_1,x_2,x_3) &= x_1+x_2+x_3 \\ E_2(x_1,x_2,x_3) &= x_1x_2+x_1x_3+x_2x_3 \\ E_3(x_1,x_2,x_3) &= x_1x_2x_3 \end{aligned}$$

By convention, $E_0(x_1,\ldots,x_n)=1$.

In general, the k-th symmetric polynomial in n variables is the following sum of all possible products of k of the n variables:

$$E_k(x_1,\ldots,x_n)=\sum_{1\le j_1<j_2<\cdots<j_k\le n} x_{j_1}\cdots x_{j_k}.$$

(this notation means that a sum is taken over all possible terms for which $j_1,\ldots,j_k$ satisfy $1\le j_1<j_2<\cdots<j_k\le n$).

Show that

$$E_k(x_1,\ldots,x_{n+1})=E_k(x_1,\ldots,x_n)+x_{n+1}E_{k-1}(x_1,\ldots,x_n).$$

Problem 20. [Harder] Derive the following formula for $\tan(\alpha_1+\cdots+\alpha_n)$. Show that all n,

$$\tan(\alpha_1+\cdots+\alpha_n)=\frac{E_1^{(n)}-E_3^{(n)}+E_5^{(n)}-\cdots}{E_0^{(n)}-E_2^{(n)}+E_4^{(n)}-\cdots}$$

where $E_k^{(n)}=E_k(\tan\alpha_1,\ldots,\tan\alpha_n)$ are the elementary symmetric polynomials defined in the previous problem. Check that these formulas are compatible with the formulas you found for $\tan(\alpha+\beta)$ and $\tan(\alpha+\beta+\gamma)$.

Problem 21. [Harder] Show that

$$1+2\cos\alpha+2\cos 2\alpha+2\cos 3\alpha+\cdots+2\cos n\alpha=\frac{\sin(n\alpha+\alpha/2)}{\sin(\alpha/2)}.$$