The game "Nim"
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Today we will play the game called "Nim". The name of the game comes from the German word "Nimm!" which means "take". After a lot of experimentation, we will learn that actually there is a beautiful mathematical theory of this game, which allows you to predict who wins the game if everyone plays their best.

Rules of the game:

- *The game is played by two players; Players alternate in making the moves.*
- *The game starts when several piles of chips are placed on the table. For example, there could be three piles of 3, 4 and 5 chips respectively.*
- *A move consists of selecting a pile and removing any number of chips from it. You must take at least one chip. You may remove one chip, or all chips, or any intermediate number of chips from a single pile.*
- *The player who takes the last chip wins.*

Comments:

- You may NOT remove chips from more than one pile in one move. However, from the pile you have selected, you can remove any number of chips (even all of them).
- Sometimes, people play a version of this game where the player who has to take the last chip looses. For now, we will only deal with the version where the player who takes the last chip wins.
- When playing the suggested positions of the game with your partner, you might want to alternate being the first and the second player.

Notation: Let's use a shorthand notation to denote position in this game. The notation lists the current numbers of chips in each pile. For example, $(4, 5, 7)$ denotes a game position with 3 piles, of sizes 4, 5 and 7 respectively. Here is another example: $(2, 11, 1, 4)$ denotes a position with 4 piles of sizes 2, 11, 1 and 4 respectively.

1 Playing the Game

1. First, let's play this game with just one pile. Put several chips in a row to represent a pile. How would the game go? (Think for a second before looking below)

   The first player can always win by just taking the whole pile! That's not a very interesting game. So, let's increase the number of piles.

2. Consider the game with two piles, where both piles have the same number of chips. E.g., take two piles of size 5 each. (In our notation, the initial position is $(5, 5)$).
   Play this game with your partner several times. You can use different starting number of chips each time, as long as you have the same number in each of the two piles. Discussing your experience, can you predict which player (First or Second) can always win from this starting position.
   (Use the space below for your notes. Then write down your conclusion).

"Return to balance": As you have discovered, if the initial position is of the form (n, n) (that is, there are two piles, each of size n), then the second player has a winning strategy.
Here is a nice way to summarize this strategy:

Move to a position with equal number of chips in the two piles.

For example, if you start with $(5, 5)$ and the first player moves into position (5,3) by taking two chips from the second pile, the second player should remove two chips from the first pile to move to $(3, 3)$.

Player making the move	Resulting position
initial position	$(5, 5)$
1	$(5, 3)$
2	$(3, 3)$
1	$(0, 3)$
2	$(0, 0)$

and the second player wins!

3. Convince yourself that player two always wins using this strategy if the starting position is of the form (n, n). (Hint: convince yourself that player 1 cannot win!)

4. Now experiment with an initial position in Nim which has just 2 piles, but the number of chips in one of the piles is by 1 bigger than the number of chips in the other. (E.g., start with $(5, 6)$ or $(3, 4)$ or a similar position).
   Try to use what we have learned about the (n, n) position to show that for the initial position $(n, n+1)$ the first player has a winning strategy. (Hint: reduce to the case of two piles with equal number of chips).

5. Here is another initial position: let's take 3 piles, two of which have equal number of chips, and the third has an arbitrary number of chips (could be the same or different from the first two piles).
   E.g., for the purposes of playing, you can start with $(2, 7, 7)$ or $(4, 3, 3)$.
   Play this game several times with your partner. Which player do you think can win now? How?

6. Let the initial position consist of several piles with each pile having just one chip. (E.g., you can take initial positions such as $(1,1,1)$ or $(1,1,1,1)$). Which player wins from such an initial position? Does the answer depend on the number of piles?

7. Just to see that things can become more complicated very quickly, can you figure out who wins from the initial position $(2,4,7)$? What about $(1,2,3,4)$?

2 Writing numbers in binary (or base 2) notation

We saw that without a lot of experimentation and a complicated analysis, it is hard to predict who can win the game of Nim with a given initial position.
It turns out that there is a complete mathematical theory of how to predict who can win and how to find a winning strategy.
First, we need to start with learning the binary notation for numbers. To do this, recall how we represent numbers in decimal notation first.

2.1 What does decimal notation mean?

Consider the decimal notation for $N = 6503$

$$6503 = \mathbf{6} \times 10^3 + \mathbf{5} \times 10^2 + \mathbf{0} \times 10^1 + \mathbf{3} \times 10^0.$$

This is what is known as *positional notation*: the value of each digit in the decimal notation for x varies with the position of the digit:

$$\begin{array}{cccc} 6 & 5 & 0 & 3 \\ \uparrow & \uparrow & \uparrow & \uparrow \\ 10^3 & 10^2 & 10^1 & 10^0 \end{array}$$

Thus the value of each digit increases by a factor of 10 as we move right to left. Very soon we'll use a different system of notation in which the role of 10 is played by 2. To indicate that we are dealing with the decimal notation, we'll write $N = (6503)_{10}$.

2.2 What does the binary notation mean?

In binary (or "base 2") notation, only digits 0 and 1 are used. As in decimal notation, each digit again has a different value depending on its position. This value increases by a factor of 2 (rather than 10) when we move to the left. We'll use the subscript 2 to indicate binary notation.

To say that $N = (1101)_2$ means that

$$N = \mathbf{1} \times 2^3 + \mathbf{1} \times 2^2 + \mathbf{0} \times 2^1 + \mathbf{1} \times 2^0.$$

Thus

$$N = 1 \times 8 + 1 \times 4 + 0 \times 2 + 1 \times 1 = 13.$$

The value of each digit is indicated below:

$$\begin{array}{cccc} 1 & 1 & 0 & 1 \\ \uparrow & \uparrow & \uparrow & \uparrow \\ 2^3 & 2^2 & 2^1 & 2^0 \end{array}$$

- Fill in the following table:

Binary notation for N	Expression for N	Value of N
1101	$1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$	13
10		
100		
1111		

The table below summarizes various powers of 2 and may come in handy:

2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7
1	2	4	8	16	32	64	128

2.3 Finding binary notation for a number

2.3.1 How many binary digits does a number N require?

Given N, find the largest power of 2 that fits into N (i.e., is no larger than N). For example, if $N = 11$, the largest power of 2 that fits into N is $8 = 2^3$. The number of digits required is then one more than this power (4 in this case).

- Fill in the following table:

N	Largest power of 2 that fits into N	Number of binary digits
11	$8 = 2^3 \leq 11$	4
9		
13		
15		

2.3.2 Finding the binary notation for a number N

Now that we know how many digits N has, we also know the left-most digit (it's 1, the only nonzero digit we have).

For example, we therefore know that $11 = (1???)_2$ since 11 requires 4 digits. Now subtract from N the largest power of 2 that fits: $11 - 2^3 = 11 - 8 = 3$. Find the largest power of two that fits into that number ($2^1 = 2$ fits into 3). This gives you that the next non-zero digit is the one corresponding to 2^1: $11 = (101?)_2$. Now subtract from 3 the largest power of two that fits into it ($3 - 2^1 = 3 - 2 = 1$) and find the largest power of 2 that fits into that number ($2^0 = 1$). This gives you that the next nonzero digit is the one corresponding to 2^0: $11 = (1011)_2$.

Fill in the following table:

N	Power of 2 fitting into N	Remainder	Power of 2 fitting into remainder	Remainder	Power of 2 fitting into remainder	Remainder	Power of 2 fitting into remainder	Binary notation
11	$8 = 2^3$	$11 - 8 = 3$	$2 = 2^1$	$3 - 2 = 1$	$1 = 2^0$	—	—	1 0 1 1 $\uparrow$ $\uparrow$ $\uparrow$ 2^3 2^1 2^0
13								
15								
9								

3 Using binary notation to solve Nim

3.1 Binary notation for Nim

Now, using binary notation, we can convert a Nim-game into a game which can be analyzed mathematically.

1. First, for every pile of chips in a Nim-position, rewrite the number of chips using binary notation. For example, for a pile consisting of 5 chips you get $(101)_2$.

2. Then, write all these binary numbers one under the other to get something that looks like a table (with no dividers).
   For example, for a 3-pile position $(1, 2, 5)$ you will get the following table:

$$\begin{array}{ccccc} 1 & \to & 0 & 0 & 1 \\ 2 & \to & 0 & 1 & 0 \\ 5 & \to & 1 & 0 & 1 \end{array}$$

   (Note that we have put extra zeros in the beginning of the first two rows. This allows us to get a table consisting of 1s and 0s with no empty places.)
   Now, do your own conversion: for a 3-pile position $(2, 3, 4)$, what is the table that you get?

$$\begin{array}{cc} 2 & \to \\ 3 & \to \\ 4 & \to \end{array}$$

3. So far, we have converted Nim-positions to tables of zeros and ones, using binary notation. Let's see if we can convert the whole game of Nim into a new game with these tables!
   The main rule in Nim is that on a single move, a player is allowed to take as many chips from a single pile as he/she wants. Now, answer the following:

   - What part of the table corresponds to a single pile in Nim?

   - What happens to the corresponding row of the table when you take away some chips from a Nim-pile? (Try several examples before reading the next page)

4. In summary, a move in the game of Nim corresponds to:

   - First, select a row in the table and choose a 1 in that row. Change that 1 to a zero. (This ensures that some chips are taken away).
   - After that, modify some or all of the digits to the right of that 1 in an arbitrary way (it is arbitrary because you are allowed to subtract an arbitrary number).

   Here is an example: take the Nim-position $(2, 3, 4)$. Write down the corresponding table (you already found it in an earlier exercise):

$$\begin{array}{cc} 2 & \rightarrow \\ 3 & \rightarrow \\ 4 & \rightarrow \end{array}$$

   Now, modify the third row (which corresponds to the last pile) so that it becomes $(0\ \ 1\ \ 1)$. Let's figure out what is the corresponding move in Nim.

   - First, which number does the binary notation $(011)_2$ represent?

   - Now, your move corresponds to taking away some chips from the third pile. This number of chips is the difference between 4 and the number of remaining chips that you have just found. So, how many chips have you removed?

3.2 A strategy to win at Nim

Writing Nim in binary notation explains how to win at Nim.

Let us say that a Nim position corresponds to the following table:

$$\begin{array}{ccccc} 7 & \rightarrow & 1 & 1 & 1 \\ 2 & \rightarrow & 0 & 1 & 0 \\ 5 & \rightarrow & 1 & 0 & 1 \end{array}$$

Note that *each column has an even number of* 1*'s.* Such a position is called *balanced.*

- If the game is at a balanced position, the next move results in an unbalanced position. Explain why this is so. (Remember that with each move there will be a 1 in some column changed to a zero).

You win if you can put the game in a balanced position. Here is why. Let us say that the game is in a balanced position, and you opponent moves in some way. After his move, the position must be unbalanced. Then *you can always make a move to put the game into a balanced position.*

For example, if the your opponent took away 4 chips from the first pile, the position would be

$$\begin{array}{ccccc} 3 & \rightarrow & 0 & 1 & 1 \\ 2 & \rightarrow & 0 & 1 & 0 \\ 5 & \rightarrow & 1 & 0 & 1 \end{array}$$

which is not balanced (the leftmost column has only a single 1). To balance this position, you should get rid of a 1 in the leftmost column, which we can do by modifying the last row:

$$\begin{array}{ccccc} 4 & \rightarrow & 0 & 1 & 0 \\ 2 & \rightarrow & 0 & 1 & 0 \\ 1 & \rightarrow & 0 & 0 & 1 \end{array}$$

and the position is again balanced.

With each move, fewer chips will be left. But your opponent can never take the last chip: the position with no chips is a balanced position (all rows are zero!) and since he is always forced to start at a balanced position, there is no way that he can get to a balanced position!

- Now use this strategy to play Nim!

3.3 Examples

1. Consider the initial position $(2, 4, 7)$. Convert each of the numbers into the binary notation: Is this

   position balanced or unbalanced? Which player can win and how?

2. Analize the initial position $(1, 2, 3, 4)$ in a similar way.