INEQUALITIES
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A. Quadratic inequalities

Let us consider the well known quadratic equation

$$ax^2 + bx + c = 0$$

Its solutions are given as

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

provided that we have $b^2 - 4ac \geq 0$.

By the way, the quadratic equation can also be written as

$$x^2 + px + q = 0$$

with solutions $x_{1,2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$. This is a common formula used in European textbooks. Exercise: show that the abc formula can be rewritten as the pq formula and vice versa, Which condition on p,q ensures that solutions exist over the real numbers?

Back to our quadratic inequalities. We could ask for all x which satisfy

$$ax^2 + bx + c > 0$$

After factorization on the left hand side, the inequality becomes

$$a(x - x_1)(x - x_2) > 0$$

So for $a > 0$ the solution is the range of values for which either ($x - x_1 > 0$ and $x - x_2 > 0$) or ($x - x_1 < 0$ and $x - x_2 < 0$).

These observations are very intuitive when done graphically. However, even for problems involving only parabolas, the algebra can lead to tricky expressions and may involve branching into carefully chosen cases.

Sometimes, one can use absolute value notation to simplify or emphasize steps in the process. For the quadratic equation this might be used when $x_1 = x_2$ since then $(x - x_1)(x - x_2) = (x - x_1)^2$ and we write $(x - x_1)^2 > 0$ equivalently as $|x - x_1| > 0$ based on the definition

$$|x| = \sqrt{x^2}$$

Equivalence is important here, and logical mistake may result if we don't assure that steps work in two directions. For example take this sequence of steps

$$x-1<\sqrt{2x+1} \Longrightarrow (x-1)^2<2x+1 \Longrightarrow x^2<0$$

There is no solution to the last inequality, but $x=0$ works in the first inequality, so you can not derive a solution at the end of these steps. Or take

$$x^2>4 \Longrightarrow x>2$$

This implication is false and should read

$$x^2>4 \Longrightarrow |x|>2,$$

because we also have $x<2$ satisfying the first expression. Also, take a look at

$$x>2 \Longrightarrow x^2>4$$

This is a true statement, but the right expression does not have the same solution as the left expression (it has more) since the converse of this implication is not true.

Now it is time for you to look at a selection of introductory problems. In each case, try to illustrate with a graph.

Problems:

1. Determine the solution of $x^2-4x+1>1$.
2. Find the domain of the function $\ln\left(\frac{x-1}{x+1}\right)$.
3. For which x do we have $x<x^3$?
4. Solve inequality $x^2-6x+5>4x^2-7x+3$.
5. Show that for all $x>0$ we have $x+\frac{1}{x}\geq 2$.

B. Quadratic inequalities with parameters

Solve inequality $2x^2 + ax + 2 < 0$.

Since the coefficient near x^2 is positive we immediately see that if the discriminant $D = a^2 - 16 \le 0$ we have no solutions. If $D = a^2 - 16 > 0$ then the solution is

$$\frac{-a - \sqrt{D}}{4} < x < \frac{-a + \sqrt{D}}{4}$$

Therefore, when $a \in (-\infty, -4) \cup (4, \infty)$ we obtain

$$\frac{-a - \sqrt{a^2 - 16}}{4} < x < \frac{-a + \sqrt{a^2 - 16}}{4},$$

and there are no solutions when $a \in [-4, 4]$.

Problems:

1. Solve the inequality $ax^2 - 2x + a > 0$.

2. For which a do we have $x^2 + (3a + 4)x + a^2 > 0$ for all x?

3. Solve the inequality $ax^2 - 2(a + 1)x + 1 > a$.

C. Inequalities with square roots

Solve the inequality $\sqrt{x^2+x-2} \geq x+3$.

The problem doesn't state the domain of the functions involved. This is typical, and we need to make sure all restrictions on x are accounted for. Here, we need $x^2+x-2 \geq 0$, so $(x+2)(x-1) \geq 0$ which means $x \geq 1$ or $x \leq -2$.

Remember that a square root is never negative, and hence it makes sense to split this problem into the cases $x+3 < 0$ and $x+3 \geq 0$.

In the first case, our inequality is always true as long as x is in the domain of the original left side. So $x < -3$ works all the time since all $x \leq -2$ are in that domain.

In the second case, we can square both sides to equivalently obtain

$$x^2+x-2 \geq (x+3)^2$$

and simplify to $5x+11 \leq 0$, i.e. $x \leq -\frac{11}{5}$. Together with the assumption of case two we obtain $-3 \leq x \leq -11/5$. Note that $-11/5 < -2$, so these values are in the domain of the expression under the square root.

Putting all the pieces together, our solution is

$$-\infty < x \leq -11/5.$$

Problems:

1. $\sqrt{3x^2+22} < 2x+3$
2. $\sqrt{x+1}+\sqrt{x+2} > 2$
3. $\sqrt{13-3x} \geq x-1$
4. $\sqrt{x^2+3} \leq x^2+1$

D. Inequalities with absolute values

Solve the inequality $|x-5|+|x+1| \geq x+3$.

We split into three cases.

For $x \leq -1$ we get $-(x-5)-(x+1) \geq x+3$ and hence $x \leq 1$ from here, but the condition in this case is already $x \leq -1$.

For $-1 < x \leq 5$ we get $-(x-5)+(x+1) \geq x+3$ and hence $-1 < x \leq 3$.

For $x > 5$ we get $(x-5)+(x+1) \geq x+3$ and hence $x \geq 7$.

Altogether, this means

$$-\infty < x \leq 3 \quad \text{or} \quad x \geq 7$$

Alternatively, we could have observed that absolute values are not negative, and hence the inequality is for sure true for all $x \leq -3$. Subsequently cases $-3 < x \leq 5$ and $x > 5$ are treated as in the previous solution.

Problems:

1. $|x-1| > 1$

2. $|2x+4| - x \leq |x-6| - 2$

3. $|x^2+3x-10| - 3x \geq 6$

E. Challenge problem

Suppose that a, b, c, A, B, C are real numbers, $a \neq 0$ and $A \neq 0$, such that

$$|ax^2+bx+c| \leq |Ax^2+Bx+C|$$

for all real numbers x. Show that

$$|b^2-4ac| \leq |B^2-4AC|.$$