NOTES ON DIMENSION.

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1. Subsets of [0,1].

Recall that any real number can be written using decimal notation, e.g.:

$$x = 2.718281828459045\ldots$$

We call the string of digits $2.71828\ldots$ the *decimal representation* of the number x. For the rest of the day we'll be dealing with numbers that are in $[0,1)$ and so the first digit will always be zero; for example, $x = 0.1000000\ldots = 1/10$, $x = 0.33333\ldots = 1/3$.

Problem 1. We say that a decimal representation of x is *periodic* if, after removing some finite number of digits from the beginning of the decimal representation of x, what is left is a repeating string. For example, $0.123333\ldots$ and $0.12343434\ldots$ are both periodic.

Show that any rational x has a periodic decimal representation. Conversely, show that any x that has a periodic decimal representation is rational. (Recall that x is called rational if $x = a/b$ where a, b are integers; in other words, x is rational if nx is integer for some integer n).

Problem 2. Let s be a string of digits, each digit chosen from among $\{0,\ldots,9\}$, e.g. $s = 12345\ldots$.

(a) Fix n, and consider the set X_n of all numbers x for which the first n digits (after the decimal point) equal to the first n digits of s. For example, if s is as above and $n = 3$, X_3 is the set of all numbers of the form $0.123\ldots$, where the digits after 3 are arbitrary. Describe the set X_n.

(b) Show that as n increases, the sets X_n decrease.

(c) The *nested interval principle* states that if I_n is a collection of closed intervals, so that I_{n+1} is a subset of I_n, then the intervals I_n must have a common point. Conclude that *all* the sets X_n, $n = 1, 2, 3, \ldots$ have a point in common.

(d) Show that there can be no other point in common between all the X_n (hint: what is the length of X_n?)

The point you found in this problem has a decimal representation $0.s$.

Because of part (d) of problem 2, you see that any infinite sequence of digits s determines a number x in $[0,1]$. If we denote by s_n the n-th digit of s, you can think of the number as being an "infinite sum":

$$x = s_1 10^{-1} + s_2 10^{-2} + s_3 10^{-3} + \cdots$$

(indeed, this is exactly right if all but a finite number of digits in s are zero; and that is exactly what we usually mean by a decimal representation). There are several ways of making sense of this sum when there is an infinite number of digits, one of such ways being what you did in Problem 2.

There is a surprise waiting for us, as we'll shortly see:

Problem 3. Can two *different* strings s and s' give rise to the same number? *Hint:* take a look at $0.9999\ldots$.

You can in fact characterize when this happens exactly:

Problem 4. Show that the only time two strings s and s' give rise to the same number is when (up to switching s and s'), $s = a_1 \ldots a_n 9999\ldots$ and $s' = a_1 \ldots a_{n-1} a'_n 0000\ldots$ where $a'_n = a_n + 1$. In other words, the only numbers that have an ambiguous decimal representation are ones for which there is a decimal representation involving only a finite number of non-zero digits.

Because of Problem 2, we cannot talk about *the* decimal representation of a number, because there may be two such representations. However, because of problem 4, we can make a choice of such representations: we'll prohibit decimal representations that have an infinite tail of 9's. With this convention, we have concluded that *every number in [0,1] has a unique decimal representation which does not have an infinite tail of* 9*'s.*

Problem 5. Describe the set Y consisting of all numbers x in $[0, 1]$ so that in their decimal representation the digit 5 never occurs. *Hint:* first find the set Y_n consisting of those numbers x for which the digit 5 does not occur among the first n digits of x.

The set Y is called a *Cantor set.*

Problem 6. What is the "length" of Y? *Hint:* show that this length can be as little as possible by representing Y_n as the result of deleting several disjoint intervals from $[0, 1]$, and find out the length of what you deleted.

2. Some notions of dimension.

The main question we aim to address in this section is:

> How can we tell the dimension of a set? For example, certain objects are one-dimensional (e.g. line segments in 3D space), or two dimensional (e.g. surfaces in 3D) or three-dimensional. How can this be detected?

2.1. Minkowski content.

Let X be a subset of $2D$ space, and let's assume that X is "bounded" (i.e., it fits inside a disk of a sufficiently large radius).

We'll denote by $N_t(X)$ the t-neighborhood of X, i.e. $N_t(X)$ is the set of all points which are distance at most t from some point of X.

Example 1. If X is a single point, $N_t(X)$ is a disk of radius t centered at that point. If X consists of two points, then N_t consists of the two disks each of radius t centered at those two points.

Problem 7. In each of the following cases: (a) X a single point; (b) X a line segment; (c) X a disk, find a formula for the area of $N_t(X)$.

The idea is to now look at the rate at which the area shrinks as t decreases to zero.

Problem 8. (a) Assume that $f(t) = t^d$. Find an expression for

$$\frac{\log f(t)}{\log t}.$$

(b) Assume now that $f(t) = Ct^d$ for some fixed C. Show that for small t,

$$\frac{\log f(t)}{\log t}$$

is very close to d.

(c) Guess a formula for the dimension of X in terms of the areas of $N_t(X)$.

This formula is called the *Minkowski dimension* of X.

Problem 9. Redo everything in this section for a subset X' of $3D$ space. Explain why if X' is contained in a plane, then its dimension (measured using areas in that plane) is the same as the one measured using volumes in three dimensions. Now redo everything for a subset X'' of 1-dimensional space.

2.2. **Packing and covering dimensions.** Assume now that X is a subset of the line and that X is bounded. Denote by $K_t(X)$ the smallest number of intervals each of length t needed to cover X. Denote by $P_t(X)$ the largest number of intervals of length t so that the intersections of these intervals with X are disjoint and nonempty.

Problem 10. Let X be (a) a point; (b) a line segment. Find $K_t(X)$.

Problem 11. Show the following:

$$tP_t(X) \leq \text{Area of } N_t(X) \leq tK_t(X).$$

Problem 12. Show that $P_t(X) \geq K_{2t}(X)$. *Hint.* Assume that X is covered by p intervals of length t, and the number p is minimal, i.e., $p = P_t(X)$. Now replace each interval with another interval of twice the length, but centered at the same point. Show that these intervals must cover X (if they don't, was p really minimal?)

Problem 13. We thus have (for $t < 1$, so that $\log t < 0$):

$$\frac{\log tP_t(X)}{|\log t|} \leq \frac{\log \ \text{Area of } N_t(X)}{|\log t|} \leq \frac{\log tK_t(X)}{|\log t|} \leq \frac{\log tP_{t/2}(X)}{|\log t|}.$$

Replace t by $2s$ in the last equation and notice that

$$\frac{\log 2sP_s(X)}{|\log 2s|} = \frac{\log sP_s(X) + \log 2}{|\log s + \log 2|} \approx \frac{\log sP_s(X)}{|\log s|}$$

for very small s. Conclude that for t extremely small, all of the numbers

$$\frac{\log tP_t(X)}{|\log t|}, \quad \frac{\log tK_t(X)}{|\log t|}, \quad \frac{\log \text{Area of } N_t(X)}{|\log t|}$$

are approximately the same. How does the last formula compare to the one you found in the previous section?

Definition 1. If for very small t, the quantity $\frac{\log tK_t(X)}{|\log t|}$ becomes approximately equal to a number d, we call d the covering (or packing) dimension of X.

3. The dimension of a Cantor set.

Problem 14. What is the covering dimension of the set Y you found in Problem 5? *Hint: estimate the covering and packing numbers of Y.*

Instead of working this out for all t, let us consider the case that $t = 10^{-m}$. Consider the set Y_m consisting of numbers none of whose first m digits are equal to 5. Then Y is contained in Y_m and moreover Y_m itself can be covered by 9^m intervals of length 10^{-m}. On the other hand, these same intervals are disjoint and their intersections with Y_m are nonempty, and so $K_t(Y) = P_t(Y) = 9^m$ (if $t = 10^{-m}$).

Now compute:

$$\frac{\log K_t}{|\log t|} = \frac{\log 9^m}{|\log 10^{-m}|} = \frac{\log 9}{\log 10}.$$

This is the dimension of the Cantor set.

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