Parity

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Parity is the property of an integer to be even (i.e. divisible by 2) or odd (i.e., not divisible by 2). For example, 6 is an even number and 7 is an odd number.

If among two numbers both are even or both are odd they are said to have the same parity. For example, 6 and 14 have the same parity; 7 and 11 also have the same parity.

Parity comes up in a variety of problems in different areas of mathematics.

1. Prove that the sum of two odd numbers is even and the product of two odd numbers is odd.

2. Show that if the sum of two integers is odd then their product is even.

3. Given two integers a and b, consider the expression $(a - b) \cdot a \cdot b$. Can you determine if it is even or odd?

4. Can you find 7 odd numbers whose sum is 100?

5. The product of 22 integers (i.e., whole numbers) is equal to 1. Show that their sum can not be 0.

6. The numbers 1 through 10 are written in a row. Can the signs " $+$ " or " $-$ " be placed between them, so that the value of the resulting expression is 0?

7. John writes the numbers 1 through 49 on paper cards. After that, he turns the cards over, mixes them up and writes the same numbers on the other side. For every card, John adds the numbers written on both of its sides and then multiplies all these sums. Show that the number he gets is even.

8. A group of Kindergarten students (boys and girls) dances in a circle in such a way that for every child both neighbors are of the same gender. Given that there are 10 boys in the circle, how many girls are there?

9. Ada has several pennies, nickels and dimes in her savings. She took 10 coins and counted their total, which happened to be 25 cents. Show that at least one of the coins Ada took is a dime.

10. Seven gears are placed on a plane, arranged in a circular chain. Can all the gears rotate simultaneously?

11. Can a knight start at square $a1$ of chessboard and go to square $h8$ visiting each of the remaining squares exactly once on the way. (Note that every time a knight moves from a square of one color, black or white, to a square of the other color).

12. On a chessboard, a knight starts from square $a1$, and returns there after making several moves. Show that the knight makes an even numbers of moves.

13. 100 checkers are placed in a row. On each move one can exchange any pair of checkers that has exactly one checker between them. (E.g., one can exchange 1st and 3rd checkers, or 5th and 7th, etc.) Can one reverse the order of checkers by performing this operation several times?

14. A cube has all vertices and centers of faces marked. The diagonals of each of the sides are also drawn. Can one go along the (parts of) diagonals and visit each of the marked points exactly once?

15. A King is placed somewhere on the chessboard. Two players take turns moving the King around the chessboard. According to chess rules a King can only move to an adjacent square. In addition, in this game the King is not allowed to be returned back to the square it just came from. Which player (the starting player or his opponent) can always guarantee a win in this game? Describe the winning strategy.

Problems are taken from

1. D. Fomin, S. Genkin, I. Itenberg "Mathematical Circles (Russian Experience)"

2. A.V. Spivak "A mathematical circle (grades 6–7)" (in Russian).