Logarithm Problems

1. How many positive integers b have the property that $\log_b 729$ is also a positive integer?

   $$(\mathbf{A})\ 0 \qquad (\mathbf{B})\ 1 \qquad (\mathbf{C})\ 2 \qquad (\mathbf{D})\ 3 \qquad (\mathbf{E})\ 4$$

2. The product of all real roots of the equation $x^{\log_{10} x} = 10$ is

   $$(\mathbf{A})\ 1 \qquad (\mathbf{B})\ -1 \qquad (\mathbf{C})\ 10 \qquad (\mathbf{D})\ 10^{-1} \qquad (\mathbf{E})\ \text{none of these}$$

3. For some real numbers a and b, the equation

   $$8x^3 + 4ax^2 + 2bx + a = 0$$

   has three distinct positive roots, and the sum of the base-2 logarithms of the roots is 5. What is the value of a?

   $$(\mathbf{A})\ -256 \qquad (\mathbf{B})\ -64 \qquad (\mathbf{C})\ -8 \qquad (\mathbf{D})\ 64 \qquad (\mathbf{E})\ 256$$

4. What is the value of the sum

   $$S = \log_{10}(\tan 1^\circ) + \log_{10}(\tan 2^\circ) + \cdots + \log_{10}(\tan 88^\circ) + \log_{10}(\tan 89^\circ)?$$

   $$(\mathbf{A})\ 0 \qquad (\mathbf{B})\ \frac{1}{2}\log_{10}\left(\frac{1}{2}\sqrt{3}\right) \qquad (\mathbf{C})\ \frac{1}{2}\log_{10} 2 \qquad (\mathbf{D})\ \frac{1}{2}\log_{10} 3 \qquad (\mathbf{E})\ 1$$

5. Let $a \geq b > 1$. What is the largest possible value of

   $$\log_a \frac{a}{b} + \log_b \frac{b}{a}?$$

   $$(\mathbf{A})\ -2 \qquad (\mathbf{B})\ 0 \qquad (\mathbf{C})\ 2 \qquad (\mathbf{D})\ 3 \qquad (\mathbf{E})\ 4$$

6. Determine the value of ab if

   $$\log_8 a + \log_4 b^2 = 5 \qquad \text{and} \qquad \log_8 b + \log_4 a^2 = 7.$$