LAMC PRESENTATION: DRAWING SECTIONS IN THREE DIMENSIONS.

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1. Projections of three-dimensional figures.

One of the first difficulties with 3 dimensions is our inability to draw 3D pictures. We can only draw a two-dimensional *projection* of a 3D object.

Fortunately, there are a few simple facts that we can observe:

Fact 1. *The following hold:*

1. *The projection of a point is a point. However, several distinct points may have the same projection (imagine how this works!)*
2. *What can be the projection of a line? Convince yourself that this is a line or a point, give examples of both. Also, the projections of two parallel lines are either the same or non-intersecting (parallel).*
3. *What can be the projection of a plane? Convince yourself that this is either a plane, or a line.*

Because of property 1 above, it may happen that two non-intersecting lines have intersecting projections. (Give an example; such lines are called *skew*). Because of this, it is difficult to tell from a 2D projection whether any two given lines in 3D intersect or not.

However, because of property 3 above, we have the following crucial fact, which often allows one to conclude whether two lines intersect or not:

Theorem 1. *If two distinct lines lie in the same plane and their projections intersect in a single point, then the lines intersect in a single point. If two lines do not lie in any single plane, they do not intersect even if their projections do.*

Proof. Let us call the two lines ℓ_1 and ℓ_2, and let us denote by π the plane that contains both of them. Denote by $\bar{}$ the projection of an object. Thus $\bar{\ell_1}$ is the projection of ℓ_1 and $\bar{\ell_2}$ is the projection of ℓ_2. Since $\bar{\ell_1}$ and $\bar{\ell_2}$ intersect in a single point, either they are two intersecting lines, or $\bar{\ell_1} = \bar{\ell_2} =$ a point. In the latter case it must be that $\ell_1 = \ell_2$ since they are both lines through the same point, perpendicular to the plane onto which we are projecting. Since $\bar{\ell_1}$ and $\bar{\ell_2}$ are two distinct intersecting lines, the projection $\bar{\pi}$ of π cannot be a single line (since that line would have to contain two intersecting but distinct lines $\bar{\ell_1}$ and $\bar{\ell_2}$). Thus $\bar{\pi}$ is the whole plane. But this means that any point on $\bar{\pi}$ corresponds to a single point on π. Thus the point p at which $\bar{\ell_1}$ and $\bar{\ell_2}$ intersect must be the projection of a single point q on the plane π. But then this point q belongs to both ℓ_1 and ℓ_2 and thus to their intersection. □

Another very useful set of observations is the following:

Fact 2. *The following hold:*

1. *The intersection of any two planes is either empty (if the planes are parallel) or is a line.*
2. *Given any three distinct points, there is a unique plane through these points.*
3. *Given any two intersecting lines, which are not equal, there is a unique plane containing both of them.*
4. *The intersection of any plane and a line is either empty (if the line is parallel to the plane), the whole line (if the line lies in the plane), or consists of exactly one point.*

Problem 1. Consider the cube $ABCDA'B'C'D'$, where $ABCD$ is the bottom face and $A'B'C'D'$ is the top face. Are the lines BB' and CD skew? What about the lines AD' and CB'?

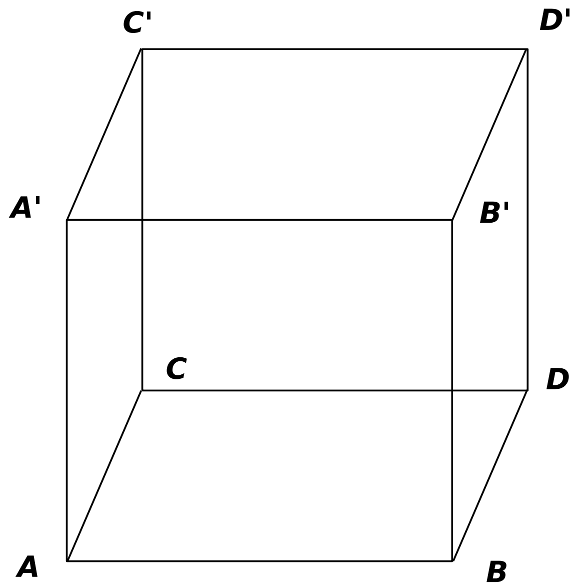
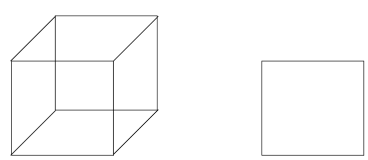


FIGURE 1. Cube $ABCDA'B'C'D'$.

2. Some remarks on drawing figures.

You may have noticed that the projection of a three-dimensional figure depends on the plane onto which we project. In other words, if you think of looking at a three-dimensional object through a glass window and tracing out what you see on the glass, the picture you get on the glass of course depends on your viewpoint.

It is also rather obvious that some projections (viewpoints) are better than others. For example, the following two drawings are both projections of a cube (see Figure 2). However, figure (a) is



(a) (b)

FIGURE 2. Projections of a cube.

clearly much better than figure (b). Indeed, many distinct edges of the cube end up being projected to the same line in figure (b).

It is important to draw pictures in which different lines project to different lines. This is always possible if one chooses the correct viewpoint:

Problem 2. Suppose that one is given n lines $\ell_1, \dots, \ell_n$ in three-dimensional space. Show that there is always a choice of a two-dimensional plane so that no two lines project to the same line.

Here are a few more hints about drawing figures:

- Draw large figures. It is OK to use up a whole page for a figure.
- Draw neatly. Draw lines straight (use a straightedge if necessary). Avoid marks and corrections that may distract you: three dimensional figures are distracting enough as they are!
- Three-dimensional figures involve a lot of intersecting lines. Some of these lines actually intersect, and some are skew. Marking the true intersections (e.g. drawing a bold dot) helps you to imagine the picture correctly.
- It may help to draw lines that are behind other planes dotted.
- It may help to use a different color when drawing supplementary lines.
- Although using a computer is sometimes helpful, the point of the exercises in this handout is to get *you* to imagine and argue about three-dimensional pictures.

3. Sections.

If S is a 3-dimensional figure (e.g., a tetrahedron or a cube) and π is a plane, then π intersects S in some way. The part of π inside of S is called the *section of S by the plane π.* The section will lie in the plane π and will in general be a polygon.

Problem 3. Let S be a (i) tetrahedron, (ii) cube. Describe all possible sections of S by planes.

3.1. **Drawing the section.** The first order of business is to find the intersection of the plane π with all of the edges of S. The idea is to proceed step by step, finding more intersections as we go.

Algorithm 1. *Let F_1 and F_2 be two faces of S intersecting along a common edge QR. Let us denote by ℓ_1 the intersection of π and F_1 and by ℓ_2 the intersection of π and F_2.*

Assume that we know ℓ_1 and that we know a single point W on ℓ_2 (see Figure 3). Then we can find ℓ_2 as follows:

1. *Extend ℓ_1 until it intersects QR. Call the point of intersection Y.*
2. *Draw a line through W and Y. The intersection of this line with the face F_2 (labeled UV in the picture) is ℓ_2.*

Problem 4. Explain why this works, i.e., why is UV the intersection of π with F_2?

Now let's use the algorithm to find the intersection of a tetrahedron with a plane. Let us suppose that the tetrahedron is as drawn in Figure 4(a) and that the plane π is the plane through the points XYZ.

It is clear that all we are looking for is one extra point: the intersection of π with the edge BC.

The faces ABD and ABC share a common edge, AB. Moreover, we know the line segment which is the intersection of π and ABD (this is the segment XZ), and also a point on the intersection of ABC and π (the point Y). Thus we can apply now algorithm. We extend XZ until its intersection with the common edge between the two sections 4(b) and call this intersection Q. Then YQ is the line along which π intersects the plane ABC. So to find the remaining intersection point, we find the intersection of YQ with BC; we call this point W.

We have now found the section: it is the quadrilateral $XYZW$.

4. More examples.

Problem 5. Consider the cube shown in Figure 5. Let π be the plane XYZ. Find the section of the cube by this plane.

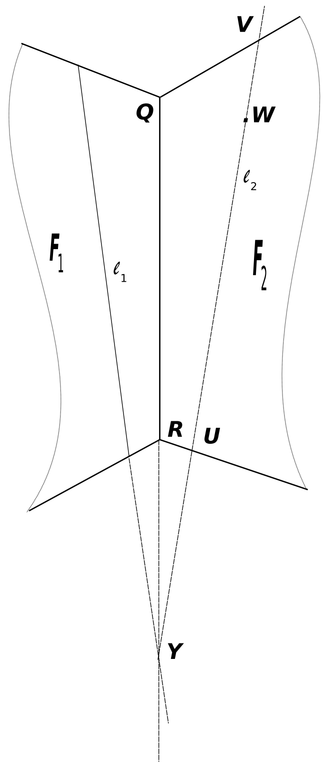


FIGURE 3. Finding the section using algorithm 1.

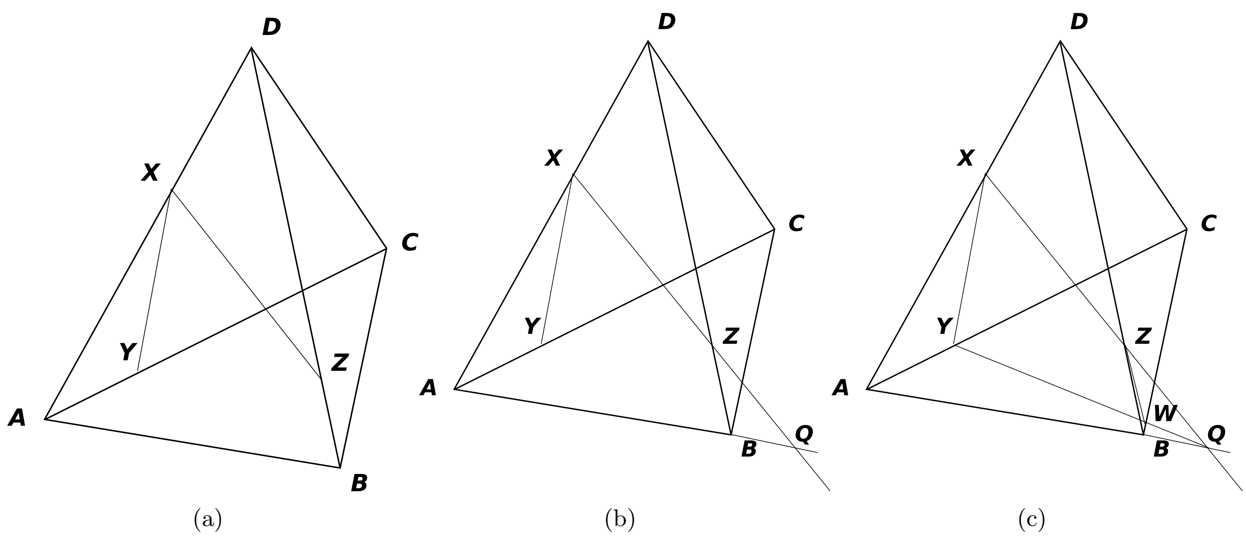


FIGURE 4. Find the section of a tetrahedron.

Problem 6. Consider the cube shown in Figure 6. Let π be the plane XYZ. Find the section of the cube by this plane.

However, sometimes we cannot use Algorithm 1, because we cannot find a single face for which we know the line at which π intersects it. Here is an example:

Problem 7. Consider the cube shown in Figure 7. Let π be the plane XYZ. Find the section of the cube by this plane.

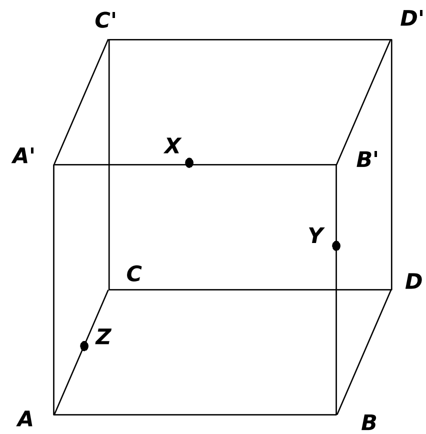


FIGURE 5. Figure for Problem 5.

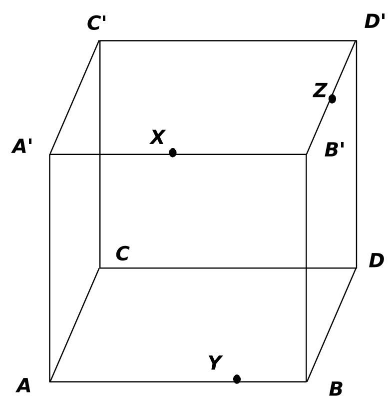


FIGURE 6. Figure for Problem 6.

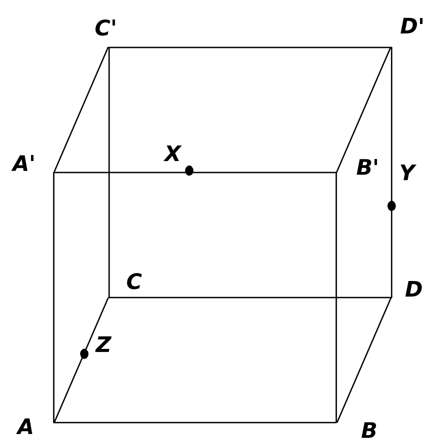


FIGURE 7. Figure for Problem 7.

5. HINTS AND SOLUTIONS.

5.1. **Hints for selected problems.** You may want to consult the hints below only after trying the problems yourself for a little while.

Hint for Problem 2. Think what happens when you change the viewpoint a little.

Hint for Problem 3. The section will be a polygon. Show that the edges of that polygon are the intersections of the plane with the faces of the polygon. This gives you an upper bound on the number of edges of the polygon. Now show that every possible number (from 3 onwards) can be achieved (and there are some more degenerate intersections, e.g. a plane intersecting at a vertex only).

Hint for Problem 5. Apply the algorithm first to the faces $ABB'A'$ and $AA'CC'$ and then to the faces $ABB'A'$ and $BB'DD'$.

Hint for Problem 7. First try to find the intersection of the plane XYZ with the bottom face of the cube. To do so, first draw a line parallel to DD' through the point X; let us denote by X' the intersection of this line with the line AB. Now because XX' is parallel to DD', $XX'YD$ all lie in the same plane (see Figure 8). Now, the lines XY and $X'D$ lie in the same plane, and thus intersect in some point Q. Show that Q is on the intersection of $ABCD$ and XYZ and then think of what that gives you (together with point Z).

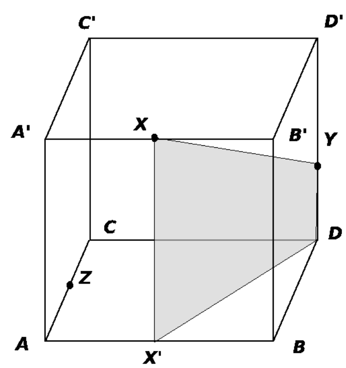


FIGURE 8. Hint for Problem 7.

5.2. **Solutions to problems.** Please look at solutions only after your have tried the problem very hard yourself! Also look at hints above.

Solution to Problem 1. The lines BB' and CD do not belong to the same plane: if they did, all 4 points B, B', C, D would be in the same plane, but C is not in the plane $BB'D$. Thus they are skew by Theorem 1.

The lines AD' and CB' belong to the same plane $ACD'B'$ (these 4 points lie the plane containing two parallel lines: AC and $B'D'$). Drawing the picture of the projection of the cube, we see that the *projections* of the lines AD' and CB' intersect, again by Theorem 1.

Solution to Problem 2. It is clear that if two lines ℓ_1 and ℓ_2 project to the same line, then this can be avoided by rotating the plane of projection by some angle α in a specific direction; and α can be chosen arbitrarily small. Moreover, it is clear that if some lines $\{\ell_k : k = i_1, \ldots, i_p\}$ have distinct projections, then there is a number α_0 so that a rotation by an angle less than α_0 in *any* direction keeps these projections distinct. Putting these statements together tells you that once some lines have distinct projections, you can move the view in such a way that they remain distinct, and the projection of one more line is distinct.

Solution to Problem 3. In the case of the tetrahedron, the section can be a convex polygon with any number of vertexes between 0 and 4. The polygon is convex because it is the intersection of two convex sets (the tetrahedron S and the plane π. Recall that a set Z is convex if together with any two points in contains the line segment joining them). 0 corresponds to a plane not intersecting the tetrahedron at all; 1 corresponds to a plane intersecting at a vertex, 2 corresponds to an intersection along an edge only. To get 3 points, take a plane parallel to the base of the tetrahedron half way to the top vertex. Figure 9 shows how to get 4 vertexes. To see that 4 is

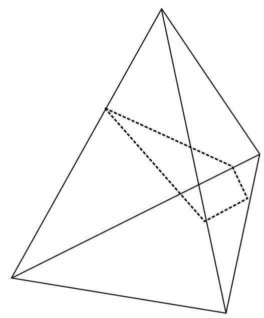


FIGURE 9. A 4-gon as a section of a tetrahedron.

maximal, note that the intersection of the plane with any face is (at most) a line segment. Thus face of S contains at most one edge of the section. Hence the number of edges of the polygon is at most the number of faces, which is 4.

The situation for a cube is similar, only one can get a convex polygon with any number of vertexes between 0 and 6.

Solution to Problem 4. The point Y belongs to ℓ_1 and thus to the plane π. Thus the line WY also belongs to the plane π. On the other hand, Y belongs to QR and so to the plane π' containing face F_2, and W belongs to F_2. Thus $YW$is contained in both π' and π, and hence must be in their intersection.

Solution to Problem 5. The solution is presented in Figure 10.

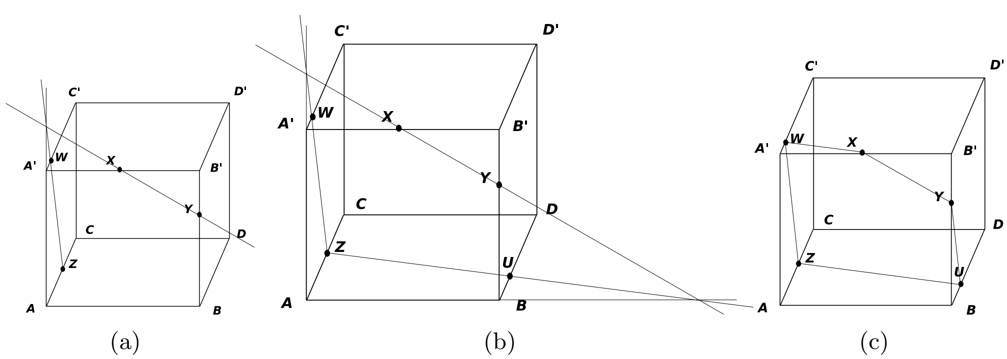
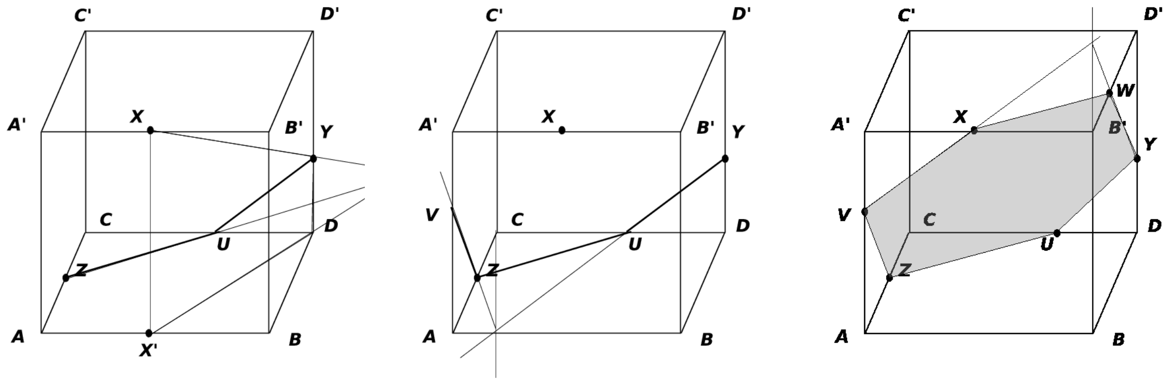


FIGURE 10. Figure for problem **5.**

Solution to Problem 7. first draw a line parallel to DD' through the point X; let us denote by X' the intersection of this line with the line AB. Now because XX' is parallel to DD', $XX'YD$ all lie in the same plane (see Figure 11(a)). Because of this, the lines $X'D$ and XY intersect in a point, Q. This point belongs to both of these lines, and thus to both the plane $ABCD$ (containing the line $X'D$) and the plane XYZ (containing the line XY). Hence ZQ belongs both to $ABCD$ and the plane XYZ. Its intersection with the bottom of the square is therefore a portion of the section of the cube by our plane $\pi = XYZ$. This gives us a point U, the intersection of the plane XYZ with the edge CD.



(a) (b) (c)

Figure 11. Solution to Problem 7.

Now we run our algorithm. Extend YU to its intersection with CC' and draw a line from that intersection through Z. This line is in the plane XYZ and also in the plane $AA'CC'$, and so its intersection with AA' gives us the intersection of XYZ with AA', V (see Figure 11(b)). Now extend VX to its intersection with BB' and draw a line from this intersection to Y. The resulting point, W, is the last point of the section (see Figure 11(c)).

6. More problems.

These problems are taken from Ch. 15 of Textbook on mathematics, G.N. Yakovlev, ed., Nauka, Moscow, 1985 (in Russian).

Problem 8. Let $ABCD$ be a tetrahedron all of whose edges have length a. The edges DA, DC and BC contain points M, N and P, respectively, so that $|DM| = |CN| = a/3$, $|CP| = a/5$ (here $|\,|$ denotes length). Find the section of the tetrahedron by the plane MNP. Denote by Q the point at which this plane intersects the edge AB. Find the length of BQ.

Problem 9. Consider the pyramid $SABCD$ with base $ABCD$. Assume that $ABCD$ is a parallelogram (i.e., opposite edges are parallel). Let M and P be the midpoints of SB and SD. Find the intersection of the plane AMP with SC. Find the ratio of the line segments into which the plane divides SC.

Problem 10. Let $ABCDA'B'C'D'$ be a parallelepiped (i.e., opposite faces are parallel). Let M, N, P be points on AA', CC' and $C'D'$, respectively, so that $|AM| : |AA'| = |C'N| : |C'C| = |C'P| : |C'D'|$. Find the point Q at which the plane MNP intersects the line BC and find the ratio $|BQ| : |BC|$.

Problem 11. Let $ABCD$ be a tetrahedron. A plane is drawn through the vertex C and the midpoints of edges AD and BD. Find the ratio of the parts into which this plane divides the line segment MN, where M and N are the midpoints of the edges AB and CD.

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