Counting vertices, edges, faces

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1 Polygons on the plane

A *polygon* is a shape on the plane with the following properties:

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Answer the following questions after experimenting with polygons with a small number of sides (triangles (3 sides), quadrilaterals (4 sides), pentagons (5 sides), etc.):

1. What happens to the number of edges if the number of vertices increases by 1?

2. What happens to the number of vertices if the number of edges increases by 1?

3. What is the relation between the number of vertices (V) of a polygon and the number of its edges (E)?

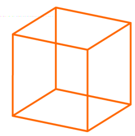
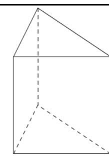
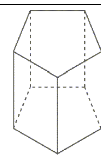
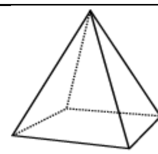
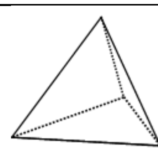
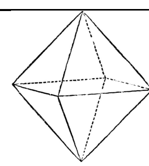
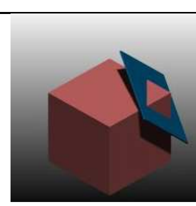
2 Vertices, Edges and Faces of Polyhedra

Complete the following definition of a polyhedron by finishing the sentences:

A *polyhedron* is an object in 3dim space with the following properties:

- polyhedron has vertices, edges and faces;
- *faces* are ;
- *edges* are ;
- *vertices* are ;
- polygons are joined along ;
- if two polygons share a part of an edge, they share the entire edge;
- all together, a polyhedron surrounds a piece of 3-dim space;

Collect data and fill in the following table:

#	Polyhedron		F(aces)	V(ertices)	E(dges)
1	cube				
2	triangular prism				
3	5-prism				
4	pyramid				
5	tetrahedron				
6	octahedron				
7	“tower”				
8	cube with a cut corner				
9	(your own)				

Explore the relations between the number of vertices, edges and faces by looking at the polyhedra in your table.

1. Is the following true or false:
   The bigger the number of faces, the bigger the number of vertices
   If it is false, find a pair of polyhedra that disproves this statement.

2. Is the following true or false:
   The bigger the number of edges E, the bigger the number of vertices V?
   If it is false, find a pair of polyhedra that disproves this statement.

3. Is the following true or false:
   The bigger the number of edges E, the bigger the number of faces F?
   If it is false, find a pair of polyhedra that disproves this statement.

4. Is the following true or false:
   The bigger the number of edges E, the bigger the sum $V+F$?
   If it is false, find a pair of polyhedra that disproves this statement:

5. Rearrange the polyhedra in the table so that the number of faces F increases:

#	Polyhedron (name)	F	V	E

Based on your findings in 4 and 5, can you make an observation about the relation of V, F and E that is true for all the polyhedra in the table?

3 Testing the conjecture

We will look at two different ways of modifying a polyhedron*: building a roof* and *cutting a corner.*

3.1 Building a roof

Take a polyhedron. We will *build a roof* over one of its faces.

- pick a face of the polyhedron and add one vertex outside of the original polyhedron, so that it is "over" the chosen face;
- connect all the vertices of the chosen face to the new vertex.

A roof over a triangular face:

1. Make a picture of a roof over a triangular face. First draw a triangle (the face). Then add a point away from the trangle. Connect all the vertices of the triangle with the new vertex.

2. Count how the numbers of vertices, edges and faces changed:

Polyhedron	Faces	Vertices	Edges
Old	F	V	E
New (roof added)			

3. Is the conjecture true for the new polyhedron?

A roof over a quadrilateral face:

1. Make a picture of a roof over a square (or any other quadrilateral) face. First draw a square (the face). Then add a point away from the square. Connect all the vertices of the square with the new vertex.

2. Fill in the table below to see how the numbers of vertices, edges and faces changed after we built the roof:

Polyhedron	Faces	Vertices	Edges
Old	F	V	E
New (roof added)			

3. Is the conjecture true for the new polyhedron?

3.2 Cutting a corner

Suppose we have a polyhedron for which our conjecture is true. Let's *cut a corner* next to one of its vertices:

- pick a vertex of the polyhedron and cut across all the edges connected to this vertex;
- remove the part that was cut off to get the new polygon.

Cutting a corner across 3 edges

1. Draw a picture of the process of cutting a corner. Draw a vertex and 3 edges coming out of it. Pick a point on each of these three edges. Cut across so that these 3 points become new vertices:

2. To see if the conjecture is true for the new polyhedron, let's see what happens with the number of vertices, edges and faces when we cut a corner:

Polyhedron	Faces	Vertices	Edges
Old	F	V	E
New (=Old without a corner)			

3. Is the conjecture true for the new polyhedron?

4. Can you explain the relation between the operations of of *building a roof* and *cutting a corner* ?

3.3 Is our conjecture always true?

To see if our conjecture is always true, we will pick a very different polyhedron.

Let's consider a very different polyhedron. For example, take a rectangular "picture frame".

1. Build a model of the rectangular "picture frame" out of Lego blocks.

2. Count the number of vertices, faces and edges and verify the conjecture. What do you get?

$$F =$$
$$V =$$
$$E =$$

3. Is your conjecture still true for this "picture frame"?

4. Can you explain how the "picture frame" is different from all the previous polyhedra we have considered?

5. Make a conclusion about the relation of the number of faces (F), vertices (V) and edge (E). Explain for what type of polyhedra it is true.

Congratulations! You have rediscovered Euler's famous formula:

$$F + V = E + 2.$$