

Math Circle  
Beginners Group  
October 25, 2015

### Warm-up problem

1. One hundred persons are lined up in a single file, facing north. Each person has been assigned either a yellow hat or a green hat. No one can see the color of his or her own hat. However, each person is able to see the color of the hat worn by every person in front of him or her. For example, the last person in line can see the colors of the hats on 99 persons in front of him or her; the first person, who is at the front of the line, cannot see the color of any hat.

Beginning with the last person in line, and then moving to the 99th person, the 98th, etc., each person is asked to name the color of his or her own hat. If the person correctly names the color of the hat, he or she lives. If the color is incorrectly named, the person is shot dead on the spot.

Everyone in the line is able to hear every response as well as the gunshot. Also, everyone in the line is able to remember all that needs to be remembered and is able to compute all that needs to be computed.

Before being lined up, the 100 persons are allowed to discuss strategy, with an eye toward developing a plan that will allow as many of them as possible to correctly name the color of their hat and thus, survive. Once lined up, each person is allowed to say only “yellow” or “green” when his or her turn arrives, beginning with the last person in line.

Your assignment: Develop a plan that allows as many people as possible to live. Do not waste time attempting to evade the stated bounds of the problem; there is no trick to the answer.<sup>1</sup>

What happens if there are  $n$  prisoners?

2. What do you think  $\max(n, m)$  means? What is  $\max(2, 3)$ ?

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<sup>1</sup>The Puzzle of 100 Hats has been sourced from The New York Times.

## Lengths of Terminating Decimals and Lengths of Periods

1. How can one find the length of a terminating decimal?

(a) Let  $\frac{a}{b}$  be a fraction in its simplest form. If  $\frac{a}{b}$  represents a terminating decimal, we know that

$$b = 2^n \cdot 5^m$$

(b) Apart from using long division, how can you convert  $\frac{a}{b} = \frac{a}{2^n \cdot 5^m}$  into a decimal? (Hint: Multiply the numerator and denominator by an appropriate number.)

i. Suppose that  $n = m$ .

$$\frac{a}{b} = \frac{a}{2^n \cdot 5^n} =$$

ii. Suppose that  $n > m$ .

$$\frac{a}{b} = \frac{a}{2^n \cdot 5^m} \times$$

iii. Suppose that  $n < m$ .

$$\frac{a}{b} = \frac{a}{2^n \cdot 5^m} \times$$

(c) How many digits are in the decimal expansion of  $\frac{a}{b}$  if

i.  $n = m$ ?

ii.  $n > m$ ?

iii.  $n < m$ ?

(d) Finally, what is the length of the terminating decimal corresponding to the fraction  $\frac{a}{2^n \cdot 5^m}$ ?

2. Find the lengths of the decimal expansions of the following fractions.

(a)  $\frac{3}{2^5 \cdot 5^3}$

(b)  $\frac{17}{1250}$

3. Sam says, “When you convert a fraction into a periodic decimal, there are only 10 possible digits in the period. Once you have used them all up in any order, you have to repeat them. So, the repeating section starts over. This means the length of the period of any repeating decimal you get from a fraction is at most 10.” Is he right? Why or why not?

(a) When you convert a fraction into a decimal using long division or the quotients method we used last week, what (if anything) forces the decimal expansion to eventually repeat itself? (Hint: The next step in the process of conversion is determined by the \_\_\_\_\_.)

(b) So, can there be a decimal expansion in which the length of the period is more than 10? Why or why not?

(c) Without calculating directly, how long could the period of the decimal expansion of the fraction  $\frac{1}{17}$  be?

(d) What is the maximum possible length of period of a number of the form  $\frac{1}{n}$ ?

## Pure and Mixed Periodic Decimals

Periods in decimals can start either immediately after the decimal point or not.

In  $0.212121\dots = 0.\overline{21}$ , the period 21 starts immediately after the decimal point. These decimals are called **pure periodic decimals**.

However, in  $0.124444\dots = 0.12\overline{4}$ , the period 4 does not immediately start after the decimal point. Such fractions are called **mixed periodic decimals**.

1. Convert the decimal  $0.\overline{124}$  into a fraction and find the prime factorization of its denominator.

2. To convert  $0.12\overline{4}$  into a fraction, write it as

$$0.12\overline{4} = 0.12 + 0.00\overline{4}.$$

- (a) Convert 0.12 (the part of the decimal before the period) into a fraction.

- (b) Convert  $0.00\overline{4}$  into a fraction as follows. Notice that  $0.00\overline{4} = \frac{1}{100} \cdot 0.\overline{4}$ . Convert  $0.\overline{4}$  into a fraction and then use this result to convert  $0.00\overline{4}$ .

- (c) Combine results in (a) and (b) to find the fraction representation of  $0.12\overline{4}$ .

- (d) Explain why the period in the decimal representation of this number does not start right away. (Hint: Think about the prime factorization of the denominators of the fractions you obtained in (a) and (b).)

3. Convert the following periodic decimals into fractions.

(a)  $0.633\dots = 0.6\overline{3}$

(b)  $0.173434\dots = 0.17\overline{34}$

(c)  $0.12312341231234\dots$

## Rational and Irrational Numbers

- The decimal number 0.123456789101112131415161718192021... is formed by writing down the digits of subsequent whole numbers. Can you convert it into a fraction? Why or why not?

A **rational number** is a number which can be written in the form  $\frac{a}{b}$ , where  $a$  and  $b$  are both integers and  $b \neq 0$ .

An **irrational number** is one which cannot be expressed as  $\frac{a}{b}$ , where  $a$  and  $b$  are both integers and  $b \neq 0$ . (It is not a number that is crazy!)

- The decimal representation of a rational number is always \_\_\_\_\_.
- Is the reverse also true? If a number has a decimal representation which is \_\_\_\_\_, does it mean that the number must be rational?
- Are the following numbers rational? Why or why not?
  - 35
  - 57.32
  - 0.1010010001...
  - 0
  - $\frac{2.735687356873568...}{3}$
  - $\frac{\frac{13}{17^2} \cdot 2^2 \cdot \frac{14}{9}}{55^3 \cdot 19}$

5. Why does the product of a rational number and a periodic decimal result in a periodic decimal?

6. Can you give another example of irrational numbers?

Pi ( $\pi$ ) is a popular example of irrational numbers.

$$\pi = 3.1415926535897932384626433832795\dots$$

There is a (very hard) proof that demonstrates that  $\pi$  is irrational.

7. Is there a fraction that equals to  $\pi$ ?

8. Another popular example of an irrational number is  $\sqrt{2}$ . Let us prove that  $\sqrt{2}$  is an irrational number.

(a) We will argue by contradiction to show that  $\sqrt{2}$  cannot be expressed as a ratio of two integers. So, we assume that  $\sqrt{2}$  is a rational number. Then in its simplest form, it can be written as

$$\sqrt{2} = \frac{a}{b}, \text{ where } a \text{ and } b \text{ do not have any common prime factors.}$$

(b) Rewrite this equality using cross-multiplication.

(c) Square both sides of the equality.

(d) Is  $a^2$  odd or even? Is  $a$  odd or even?

(e) Why is  $a^2$  divisible by 4?

(f) If  $a^2$  is divisible by 4, how do we know that  $b^2$  is divisible by 2?

(g) Is  $b^2$  odd or even? Is  $b$  odd or even?

(h) How does this contradict our original assumption?

(i) What is your conclusion?

9. Here is a visual method to prove that  $\sqrt{2}$  is irrational.

- (a) Once again, we will argue by contradiction. Let us assume that  $\sqrt{2}$  is a rational number. So, in its simplest form, it can be written as

$$\sqrt{2} = \frac{a}{b}$$

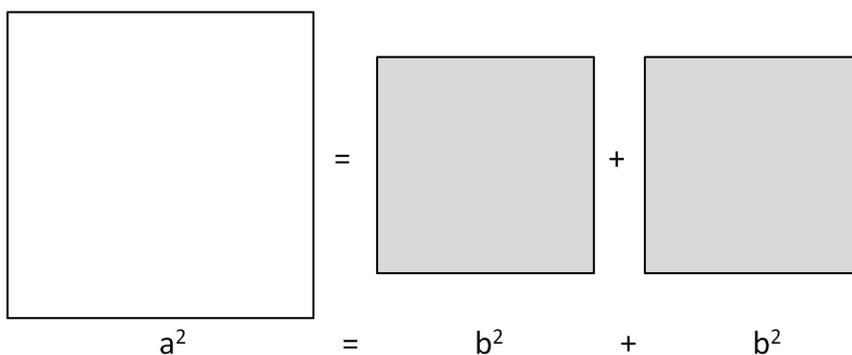
Remember that  $a$  and  $b$  are the smallest possible numbers for which this equality is true.

- (b) Rewrite this equality using cross-multiplication.

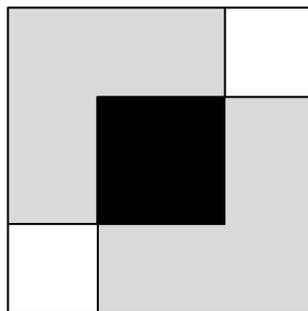
- (c) Square both sides of the equality.

- (d) Using the equality above, express  $a^2$  as a sum of two terms.

- (e) Geometrically, this means that there is a square of side  $a$  (the white square below) whose area is equal to the sum of the areas of two squares of side  $b$  (the grey squares below). Remember that we have assumed that  $a \times a$  and  $b \times b$  are the smallest possible squares.



- (f) Since  $a^2 = b^2 + b^2$ , the grey squares must fit completely inside the white square. When we try to do that, the grey squares overlap in the black square, as shown below.



- (g) Therefore, the area of the black square must be equal to the sum of the areas of the uncovered white corner squares.

$$\begin{array}{c}
 \blacksquare \\
 c^2
 \end{array}
 =
 \begin{array}{c}
 \square \\
 d^2
 \end{array}
 +
 \begin{array}{c}
 \square \\
 d^2
 \end{array}$$

The black square and the small white squares have integer sides, say  $c$  and  $d$ . Once again, we have found two numbers  $c$  and  $d$  such that  $c^2 = d^2 + d^2$ .

- (h) How does this contradict our assumption?

- (i) What is your conclusion?

10. Can you write down two numbers  $a$  and  $b$ , such that neither  $a$  nor  $b$  is rational, but  $(a + b)$  is?