The Stable Marriage Problem II

Exercise 1. Consider the preferences

Student	1	2	3	4
A	d	a	b	c
B	c	b	a	d
C	c	b	a	d
D	d	a	b	c

University	1	2	3	4
a	C	D	B	A
b	D	C	A	B
c	A	C	B	D
d	B	D	A	C

Convince yourself that the following matchings are all stable:

$$\begin{aligned}
M_1 &= \{Aa, Bb, Cc, Dd\} \\
M_2 &= \{Ab, Ba, Cc, Dd\} \\
M_3 &= \{Ac, Ba, Cb, Dd\} \\
M_4 &= \{Ab, Bd, Cc, Da\} \\
M_5 &= \{Ac, Bd, Cb, Da\} \\
M_6 &= \{Ac, Bd, Ca, Db\}
\end{aligned}$$

Draw a Hasse diagram that shows which matchings are preferred by all of the students. Some pairs M_i and M_j may be incomparable in the sense that some students prefer M_i while others prefer M_j. Are there any other stable matchings for these preferences?

Exercise 2. Consider the preferences

Student	1	2	3
A	a	c	b
B	a	b	c
C	c	a	b

University	1	2	3
a	C	A	B
b	A	C	B
c	B	A	C

Suppose the students and universities know that the student-oriented Gale-Shapley algorithm will be used to assign students to universities. Can the universities mis-report their preferences in such a way to obtain a stable matching they prefer? What preferences should they choose? What (stable) matching will they obtain?

University	1	2	3
a	__	__	__
b	__	__	__
c	__	__	__

Exercise 3. If the students mis-report their preferences can they obtain a better matching? Can the students guarantee they will always get the student-optimal stable matching? How?

Exercise 4 (Many-to-one matchings)**.** Suppose universities are allowed to admit more than one student: each university u_i has a capacity c_i which is the maximum number of students that u_i can admit. How might you modify the student-oriented Gale-Shapley algorithm to find a stable matching in this case?

Exercise 5. Consider the stable matching problem with 8 students $A, B, \ldots, H$ and three universities a, b, c with quotas 2, 2, and 4 respectively. Find a stable matching (which respects the capacities) for the preferences described below.

Student	1	2	3
A	a	b	c
B	c	a	b
C	c	b	a
D	b	a	c
E	a	b	c
F	b	a	c
G	b	a	c
H	a	b	c

Uni	Cap	1	2	3	4	5	6	7	8
a	2	C	H	B	G	A	F	E	D
b	2	B	C	A	D	H	F	G	E
c	4	A	D	F	H	B	C	G	E

Exercise 6 (Incomplete preferences)**.** Suppose that each student ranks only some universities, and that each university ranks only some students. For consistency, we assume that if a students s ranks university u, then u also ranks s. Will we always be able to find a stable matching where all students and universities are matched? (For simplicity assume that each university only admits a single student).

Exercise 7. Apply the student-oriented Gale-Shapley algorithm to the following preferences

Student	1	2	3
A	e	c	a
B	b	d	a
C	c	b	d
D	d	c	
E	d	e	

University	1	2	3	4
a	B	A		
b	C	D		
c	A	C	B	
d	B	E	D	C
e	E	A		

Is every student/university matched in the output of the algorithm? Would you consider the output stable? Why or why not?

Exercise 8. Apply the university-oriented Gale-Shapley algorithm to the same preferences as the previous exercise

Student	1	2	3
A	e	c	a
B	b	d	a
C	c	b	d
D	d	c	
E	d	e	

University	1	2	3	4
a	B	A		
b	C	B		
c	A	C	D	
d	B	E	D	C
e	E	A		

Do you obtain the same matching? Is the set of unmatched students/universities the same as before?

Exercise 9 (Preferences with indifference)**.** Suppose students are allowed to be indifferent between two or more universities. That is, a student s may rank two universities u_1 and u_2 in the same position, and be equally happy attending either. Symmetrically, the universities may be indifferent between two or more students. How could we modify the definitions of *blocking pair* and *stability* to account for this indifference?

Exercise 10. Suppose there are n students and n universities, each of which has a capacity of 1, but students' and universities' preferences may contain indifference. Is there always a stable matching (using your definition from the previous problem)? How could you find one?

Exercise 11. Consider the preferences

Student	1	2	3
A	a	b	c
B	b	a	c
C	c	b	a

University	1	2	3
a	C	B	A
b	C	A	B
c	A	B	C

Find all stable matchings. How many are there?

Exercise 12. Now consider the preferences

Student	1	2	3
A	a	b	c
B	b	a	c
C	c	$\{b, a\}$	

University	1	2	3
a	C	B	A
b	C	A	B
c	A	B	C

In C's preferences, we use $\{b, a\}$ to denote that C is indifferent between b and a. Using your definition of stability from Exercise 9 find all stable matchings. How many are there?

Exercise 13 (Incomplete preferences with indifference)**.** Consider the incomplete preferences with indifference given by

Student	1	2	3
A	e	c	a
B	b	$\{d, a\}$	
C	c	b	d
D	d	c	
E	$\{d, e\}$		

University	1	2	3
a	B	A	
b	C	B	
c	A	C	D
d	B	$\{E, D\}$	C
e	E	A	

Are the matchings you found in Exercises 7 and 8 stable with respect to these preferences? Can you find a stable matching in which every student and university is matched?

Exercise 14 (Stable roomates)**.** Suppose A, B, C and D are students at the same university. The university dormitories each house two students. The four students are friends, and wish to break into pairs to live in the dorms together. Each student has preferences over her roommate as follows

Student	1	2	3
A	C	B	D
B	A	C	D
C	B	A	D
D	B	C	A

Let $\{s_1, s_2\}$, $\{s_3, s_4\}$ be a housing assignment, where s_1 and s_2 are roommates, as are s_3 and s_4. We call such an assignment *stable* if there is no unmatched pair s_i, s_j that mutually prefer each other to their assigned roommates. Given the preferences above, is there a stable housing assignment?

Exercise 15. The preferences used in exercise 1 supported 6 stable matchings with four students and universities. Can you find preferences for four students and universities which have 7 or more stable matchings? What is the greatest number of stable matchings for an instance with four students and universities?

Student	1	2	3	4
A	__	__	__	__
B	__	__	__	__
C	__	__	__	__
D	__	__	__	__

University	1	2	3	4
a	__	__	__	__
b	__	__	__	__
c	__	__	__	__
d	__	__	__	__