

The Stable Marriage Problem

Exercise 1. Consider an instance of the stable marriage problem with 4 students (*Alice*, *Bob*, *Cathy*, and *Dan*) and 4 universities (*amherst*, *berkeley*, *caltech*, and *duke*) with the following preferences:

Student	1	2	3	4	University	1	2	3	4
<i>A</i>	<i>d</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>a</i>	<i>C</i>	<i>D</i>	<i>B</i>	<i>A</i>
<i>B</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>d</i>	<i>b</i>	<i>D</i>	<i>C</i>	<i>A</i>	<i>B</i>
<i>C</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>d</i>	<i>c</i>	<i>A</i>	<i>C</i>	<i>B</i>	<i>D</i>
<i>D</i>	<i>d</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>B</i>	<i>D</i>	<i>A</i>	<i>C</i>

Can you find a stable matching between the students and universities?

Exercise 2. Consider the same preferences as the previous exercise:

Student	1	2	3	4
<i>A</i>	<i>d</i>	<i>a</i>	<i>b</i>	<i>c</i>
<i>B</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>d</i>
<i>C</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>d</i>
<i>D</i>	<i>d</i>	<i>a</i>	<i>b</i>	<i>c</i>

University	1	2	3	4
<i>a</i>	<i>C</i>	<i>D</i>	<i>B</i>	<i>A</i>
<i>b</i>	<i>D</i>	<i>C</i>	<i>A</i>	<i>B</i>
<i>c</i>	<i>A</i>	<i>C</i>	<i>B</i>	<i>D</i>
<i>d</i>	<i>B</i>	<i>D</i>	<i>A</i>	<i>C</i>

Apply the student-oriented Gale-Shapley algorithm to find a stable matching. Is the matching you found the same as the matching from the first exercise?

Exercise 3. Again using the same preferences

Student	1	2	3	4
<i>A</i>	<i>d</i>	<i>a</i>	<i>b</i>	<i>c</i>
<i>B</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>d</i>
<i>C</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>d</i>
<i>D</i>	<i>d</i>	<i>a</i>	<i>b</i>	<i>c</i>

University	1	2	3	4
<i>a</i>	<i>C</i>	<i>D</i>	<i>B</i>	<i>A</i>
<i>b</i>	<i>D</i>	<i>C</i>	<i>A</i>	<i>B</i>
<i>c</i>	<i>A</i>	<i>C</i>	<i>B</i>	<i>D</i>
<i>d</i>	<i>B</i>	<i>D</i>	<i>A</i>	<i>C</i>

Apply the university-oriented Gale-Shapley algorithm to find a stable matching. Is the matching you found the same as the matching from the first exercise? The second exercise?

Exercise 4. Again using the same preferences

Student	1	2	3	4
<i>A</i>	<i>d</i>	<i>a</i>	<i>b</i>	<i>c</i>
<i>B</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>d</i>
<i>C</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>d</i>
<i>D</i>	<i>d</i>	<i>a</i>	<i>b</i>	<i>c</i>

University	1	2	3	4
<i>a</i>	<i>C</i>	<i>D</i>	<i>B</i>	<i>A</i>
<i>b</i>	<i>D</i>	<i>C</i>	<i>A</i>	<i>B</i>
<i>c</i>	<i>A</i>	<i>C</i>	<i>B</i>	<i>D</i>
<i>d</i>	<i>B</i>	<i>D</i>	<i>A</i>	<i>C</i>

Find a stable matching M_4 which is different from the matchings M_1 , M_2 , and M_3 you found in exercises 1, 2, and 3, respectively. Which matchings do the students prefer? Which matchings do the universities prefer?

Exercise 5. One last time, consider the same preferences

Student	1	2	3	4	University	1	2	3	4
<i>A</i>	<i>d</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>a</i>	<i>C</i>	<i>D</i>	<i>B</i>	<i>A</i>
<i>B</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>d</i>	<i>b</i>	<i>D</i>	<i>C</i>	<i>A</i>	<i>B</i>
<i>C</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>d</i>	<i>c</i>	<i>A</i>	<i>C</i>	<i>B</i>	<i>D</i>
<i>D</i>	<i>d</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>B</i>	<i>D</i>	<i>A</i>	<i>C</i>

Convince yourself that the following matchings are all stable:

$$M_1 = \{Aa, Bb, Cc, Dd\}$$

$$M_2 = \{Ab, Ba, Cc, Dd\}$$

$$M_3 = \{Ac, Ba, Cb, Dd\}$$

$$M_4 = \{Ab, Bd, Cc, Da\}$$

$$M_5 = \{Ac, Bd, Cb, Da\}$$

$$M_6 = \{Ac, Bd, Ca, Db\}$$

Draw a diagram that shows which matchings are preferred by all of the students. Some pairs M_i and M_j may be incomparable in the sense that some students prefer M_i while others prefer M_j . Are there any other stable matchings for these preferences?

Exercise 6. Consider the preferences

Student	1	2	3
<i>A</i>	<i>a</i>	<i>c</i>	<i>b</i>
<i>B</i>	<i>a</i>	<i>b</i>	<i>c</i>
<i>C</i>	<i>c</i>	<i>a</i>	<i>b</i>

University	1	2	3
<i>a</i>	<i>C</i>	<i>A</i>	<i>B</i>
<i>b</i>	<i>A</i>	<i>C</i>	<i>B</i>
<i>c</i>	<i>B</i>	<i>A</i>	<i>C</i>

Suppose the students and universities know that the student-oriented Gale-Shapley algorithm will be used to assign students to universities. Can the universities mis-report their preferences in such a way to obtain a stable matching they prefer? What preferences should they choose? What (stable) matching will they obtain?

University	1	2	3
<i>a</i>	—	—	—
<i>b</i>	—	—	—
<i>c</i>	—	—	—

Exercise 7. If the students mis-report their preferences can they obtain a better matching? Can the students guarantee they will always get the student-optimal stable matching? How?

Exercise 8. The preferences used in exercises 1 through 5 supported 6 stable matchings with four students and universities. Can you find preferences for four students and universities which have 7 or more stable matchings? What is the greatest number of stable matchings for an instance with four students and universities?

Student	1	2	3	4
<i>A</i>	—	—	—	—
<i>B</i>	—	—	—	—
<i>C</i>	—	—	—	—
<i>D</i>	—	—	—	—

University	1	2	3	4
<i>a</i>	—	—	—	—
<i>b</i>	—	—	—	—
<i>c</i>	—	—	—	—
<i>d</i>	—	—	—	—