LAMC Advanced Group

Some Induction Solutions

1. Prove that for every positive integer n,

$$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

Proof. Base Case We must show that the formula is true when $n = 1$. If we substitute 1 for n in both sides of the equation, we get

$$1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1,$$

which is a true statement. So we have proven that the formula is true when $n = 1$.

Inductive Step

$$\textit{We can assume: } 1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2} \quad (1)$$

$$\textit{We must prove: } 1 + 2 + \cdots + n + (n+1) = \frac{(n+1)(n+2)}{2} \quad (2)$$

(Note that equation (2) is simply the equation you get by replacing n with $n+1$ on both sides of equation (1))

Assuming that (1) is true, we know that:

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{(n+2)(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$

but this is exactly what we needed to prove! We're done!

□

Question before the next problem: What trick from algebra did I use to go from the second line of the last equation to the third line?

2. Prove that for every positive integer n,

$$1^2 + 2^2 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof. Base Case When 1 is substituted for n in the formula, we get

$$1^2 = \frac{1 \cdot 2 \cdot 3}{6}$$

which is true.

Inductive Step

We can assume:

$$1^2 + 2^2 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{3}$$

We must prove:

$$1^2 + 2^2 + \cdots + n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6} \tag{4}$$

Again, you should make sure you know how we got equation (4) from equation (3). Now we prove (4) using (3) in almost the exact same way we did in problem 1:

$$\begin{aligned} 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)(n+1)}{6} \\ &= \frac{(n+1)(2n^2+7n+6)}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \end{aligned}$$

This is all we needed to show, but make sure you can do all the multiplying and factoring you need to do to justify each step of the equation! □

The second problem shows how useful induction can be. The problem can be solved without induction (so can the first, look up Gauss' trick), however, that would take considerably more creativity. Solving the problem with induction did not take much creativity at all: we solved it exactly the way we solved the first problem. Some of the other problems on this handout would be impossible or at least very nearly impossible to solve without mathematical induction.

Why is induction an accepted method of proving statements about the natural numbers? To see, let's look at the first problem a little more carefully. The base case shows that $1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$ is true when $n = 1$. Since we know the formula is true for $n = 1$, the inductive step shows that it is true for $n = 2$. But now we know the formula is true for $n = 2$, so the inductive step shows that it is also true for $n = 3$. But now we know the formula is true for $n = 3$, so the inductive step shows that it is also true for $n = 4$. But now we know the formula is true for $n = 4$, so the inductive step shows that it is also true for $n = 5$. But now we know the formula is true for $n = 5$, so the inductive step shows that it is also true for $n = 6$...

I don't want to keep writing forever, but you get the idea. If I kept going long enough I could conclude that the formula was true with $n = 20,000$, or 4 billion, or 17 trillion, or any positive integer.