Induction I

Math Circle

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Suppose that you wanted to climb a reeeally tall ladder. If you could climb up to the first rung, and you knew that if you climbed to any rung, you could climb the next one, could you climb to the top of the ladder?

Induction is the formalization of this idea. If you want to prove that a statement is true for every natural number, then you can show that it is true for 1, and then show that if it is true for a number n, then it must also be true for $n+1$.

Every proof by induction follows a certain pattern. First a statement is proven true for some small, easy to prove case (usually when $n = 1$). This is called the base case. The next step is proving that if a statement is true for the $n-1$'s step, then it must be true for the n'th step as well. This is called the inductive step.

1. Show that the following formula is true:

$$1 + 2 + 3 \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

2. Also, this one:

$$1^2 + 2^2 + 3^2 \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}$$

3. Also,

$$1 \cdot 2 + 2 \cdot 3 + \ldots (n-2) \cdot (n-1) + (n-1) \cdot n = \frac{n(n+1)(n+2)}{3}$$

4. Prove or disprove: For any x which is a natural number, $x^2 + x + 41$ is always a prime number.

5. **Challenge 1** Here are a few more if you are looking for a challenge!

$$(1 + 2 + \cdots + (n-1) + n)^2 = 1^3 + 2^3 + \cdots + (n-1)^3 + n^3$$

6. **Challenge 2** And one more, if you are feeling saucy. Let F_n be the n'th Fibonacci number, so $F_1 = 1, F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$. Prove that

$$F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}$$

for $n > 2$.

Ok, ok, enough of working with stale formulas. Let's draw some pictures!

7. Suppose that you work at Clear Skies Ceramics, a factory that manufactures beautiful azure blue ceramic tiles. Your factory can make these tiles in two sizes, either in a square 1 inch by 1 inch, or in a rectangle that is 1 inch by 2 inch.

   People love your tiles, because then add a decorative flair to their bathroom trim! Suppose that one section of trim is 1 inch by 2 feet long. How many different ways can you tile the trim with your with identical 1 by 1 and 1 by 2 tiles? (Hint: Start with a smaller trim that's 1 inch by n inches for $n = 1, 2, 3, \ldots$ and see if you notice a pattern.)

8. Prove that if release a new line of tiles that are formed by gluing together one of your 1 by 1 and one of your 1 by 2 tiles into an L shape, then you can tile any wall that is 2^n by 2^n if you don't have to tile the top right 1 inch by 1 inch spot.

9. You are playing a game with your friend. Your friend puts n blue and n red chips in a circle, and then you pick a starting chip, and remove each chip in the circle going counterclockwise. You win if you can remove every chip and there are never, ever more blue chips than red chips that haven't been removed. Can you always win? If so, prove it. If not, give a counterexample.

10. Prove that if you cut the plane into pieces using a bunch of nonparallel lines, and you never have 3 lines intersect at the same point, than you can always color each piece either red or blue, such that a blue piece is never touching a red piece on a side.

11. If $8^n - 1$ always divisible by 7? If so prove it, if not find a counterexample.

12. Show that 9 always divides $4^n + 15n - 1$ for any natural number n.

13. What's wrong with the following proof that all horses are the same color?

    If there was just one horse, than certainly every horse is the same color.

    If you have more than one horse, say n of them, line them up end to end. By the inductive step, horses $1, 2, \ldots, n-1$ are the same color, and so are horses $2, 3, \ldots, n$, so all horses must be the same color.