Problems from Russian Math Olympiads

LA Math Circle (Advanced)

October 4, 2015

1. Peter exchanges stickers with his friends. For every sticker he gives someone, he gets 5 stickers back. Suppose he starts the exchange with just one sticker. How many stickers will he have after 30 exchanges?

   Solution. First, we should decide what we mean by an exchange. We will consider an exchange to be any time Peter gives away one sticker and receives 5 back. For instance, after the first exchange, Peter has 5 stickers, and if he gives away these 5, receiving 25 in return, we consider that to be 5 more exchanges, not just one.

   Peter gains four stickers as a result of each trade. He makes 30 trades, so he gains a total of $4 \cdot 30 = 120$ stickers. Therefore, he has 121 stickers after 30 exchanges.

2. Write down 7 consecutive numbers so that the digit 2 is used exactly 16 times.

   Solution. $2215, 2216, 2217, 2218, 2219, 2220, 2221$ is one example. Note that the solution is not unique: we could add any number of digits which are not two to the front of all of the numbers.

3. Three jumps of a two-headed dragon equals five jumps of a three-headed dragon. It takes a two-headed dragon the same amount of time to make four jumps as it takes a three-headed dragon to make seven jumps. Which of the dragons moves faster? Explain your answer.

   Solution. Let v_2, d_2, t_2 represent the speed, distance per jump, and time per jump, respectively, of a two-headed dragon, and let v_3, d_3, t_3 represent the speed, distance per jump, and time per jump, respectively, of a three-headed dragon. Then we are given that

$$3 \cdot d_2 = 5 \cdot d_3 \implies d_3 = \frac{3}{5} d_2$$
$$4 \cdot t_2 = 7 \cdot t_3 \implies t_3 = \frac{4}{7} t_2.$$

So,

$$v_3 = \frac{d_3}{t_3} = \frac{\frac{3}{5} d_2}{\frac{4}{7} t_2} = \frac{\frac{3}{5}}{\frac{4}{7}} v_2 = \frac{21}{20} v_2 > v_2.$$

4. Two types of ladybugs live in the magical forest: some ladybugs have 6 dots, and the rest have 4 dots each. All the ladybugs with 6 dots always tell the truth. All the ladybugs with 4 dots always lie. You met several of these ladybugs.

   - The first ladybug told you: "All of us have the same number of dots."
   - The second ladybug said: "Altogether, we have 30 dots on our backs."
   - The third ladybug said: "No! Altogether, we have 26 dots on our backs."

   The rest of the ladybugs each said that only one of those three ladybugs told the truth. How many ladybugs did you meet?

   Solution.

   Case 1. All of the other ladybugs were lying.

   Then either none of the first three told the truth, two of them told the truth, or all three of them told the truth. The second and third statements are contradictory, so it is impossible that all three told the truth. Also, if none of them told the truth, then all three are liars, as are all the others. But then the first statement is true, so this is impossible. Therefore, exactly two of the first three told the truth. If the first statement is true, then all of the ladybugs must have 6 dots since the first did. But one of the first three ladybugs lied, so this is impossible. Therefore, the first statement is a lie, and the second two are true. Again, they are contradictory. So, we conclude that Case 1 is impossible.

   Case 2. All of the other ladybugs were telling the truth. Then exactly one statement is true. It can't be the first because then all of the statements would have to be true. Therefore, if x denotes the number of ladybugs with 4 dots and y denotes the number of ladybugs with 6 dots, we either have $4x + 6y = 30$ or $4x + 6y = 26$. Additionally, we know that exactly two of the ladybugs have four dots and the rest have six. Thus $x = 2$. If $4x + 6y = 8 + 6y = 30$, then $y = \frac{22}{6} = \frac{11}{3}$ which is not an integer. Therefore $4x + 6y = 8 + 6y = 24$, from which we obtain $6y = 18$, and $y = 3$. You met two ladybugs with four dots each and three ladybugs with six dots each, for a total of five ladybugs.

5. Ben multiplied a number by 10 and got a prime number. Peter multiplied the same number by 15 and also got a prime number. Could it be that both of them did their computations correctly? Explain your answer.

   Solution. Yes it could! The number could have been $\frac{1}{5}$ since $10 \cdot \frac{1}{5} = 2$ and $15 \cdot \frac{1}{5} = 3$.

6. Solve the following riddle:

   Here is a riddle written on a cup:
   Eh *is four times as much as* **Oi,**
   Oh *is four times as little as* **Ai,**
   What do you get if you add all four of them up?

   Solution. We interpret E, h, O, i, A as distinct digits, with O, A, and E nonzero (being first digits of two digit numbers). Then we are given $Eh = 4Oi$ and $Ai = 4Oh$.

   First, notice that i and h must be even (since they are the last digits of the numbers written as 4 times another number).

   Second, Oi and Oh are both less than 25. (Otherwise, we would get a three digit number after multiplying these by 4).

   Thus, $O = 1$ or $O = 2$ are the only cases.

   If $O = 2$, the only possibilities for Oi (or Oh) are $20, 22$ and 24. 20 does not work because $20 \cdot 4 = 80$ and ends with the same digit as 20. 22 does not work because the digits have to be different. 24 does not work because we would have to have $Eh = 96$, and so $Oh = 26 > 25$, and this is not possible.

Thus, $O = 1$. Considering $10, 12, 14, 16, 18$ as the possible values of Oi and Oh. Further, 10 does not work since then $h = i = 0$, and 14 does not work since then $h = A = 6$. Therefore, only 12 and 18 are possible. Since we only have to find the sum, it does not matter which of these is Oi and Oh.

Then Eh and Ai are 48 and 72. And the sum is 150.

7. A dog and a cat are pulling a sausage in two different directions. If the dog takes a bite and run away, the cat will get 300 gr more than the dog. If the cat takes a bite and runs away, the dog will get 500 gr more than the cat. How much of the sausage will be left if each of them takes a bite and runs away?

   Solution. Let C and D denote the masses of the cat's bite and the dog's bite, respectively, in grams. Let X denote the remainder after the cat and dog have taken their bite. Now draw a picture showing what C, D, and X are! Our goal is to find X. The first sentence translates to

$$D + 300 = C + X,$$

   and the second translates to

$$C + 500 = D + X.$$

   If we add the two equations together, we obtain

$$D + C + 800 = D + C + 2X.$$

   We can cancel $D + C$ and are left with $800 = 2X$, or $400 = X$.

8. Thirteen children were sitting around the table. All of the girls agreed that they will only tell the truth to each other and will lie to the boys. All of the boys agreed that they will only tell the truth to each other and lie to the girls. One of the children said to his/her neighbor on the right: "The majority of us are boys." The neighbor told his/her neighbor on the right: "The majority of us are girls," and so on, with the last child telling the first one: "The majority of us are boys." How many boys were there at the table?

Solution. We analyze the following four cases:

Case 1. The first child is a girl, the majority of the children are girls.

Then since the first child is lying, the second is a boy. The second tells the truth so the third is a boy. The third lies the truth so the fourth is a girl. The fourth tells the truth so the fifth is a girl. Etc. So the children are arranged as

$$G, B, B, G, G, B, B, G, G, B, B, G, G.$$

But this does not make sense because then the last child should tell the first child the truth, when she in fact lies. Therefore, Case 1 is impossible.

Case 2. The first child is a girl, the majority of the children are boys.

A similar analysis to Case 1 show that the children are arranged as

$$G, G, B, B, G, G, B, B, G, G, B, B, G.$$

But counting, we see that the majority of children would then be girls, not boys. Therefore, Case 2 is also impossible.

Case 3. The first child is a boy, the majority of the children are girls.

This time the arrangement is

$$B, G, G, B, B, G, G, B, B, G, G, B, B.$$

But now we see that there are more boys than girls, when we assumed that there were more girls than boys, so Case 3 is impossible too.

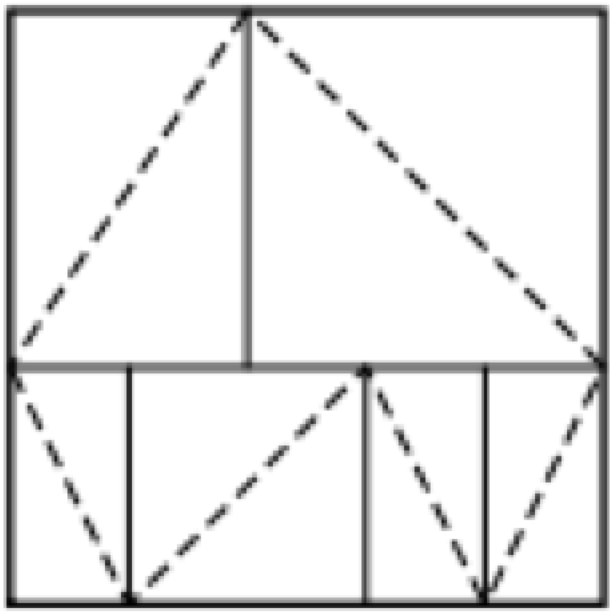
Case 4. The first child is a boy, the majority of the children are boys.

This assumption gives the arrangement

$$B, B, G, G, B, B, G, G, B, B, G, G, B.$$

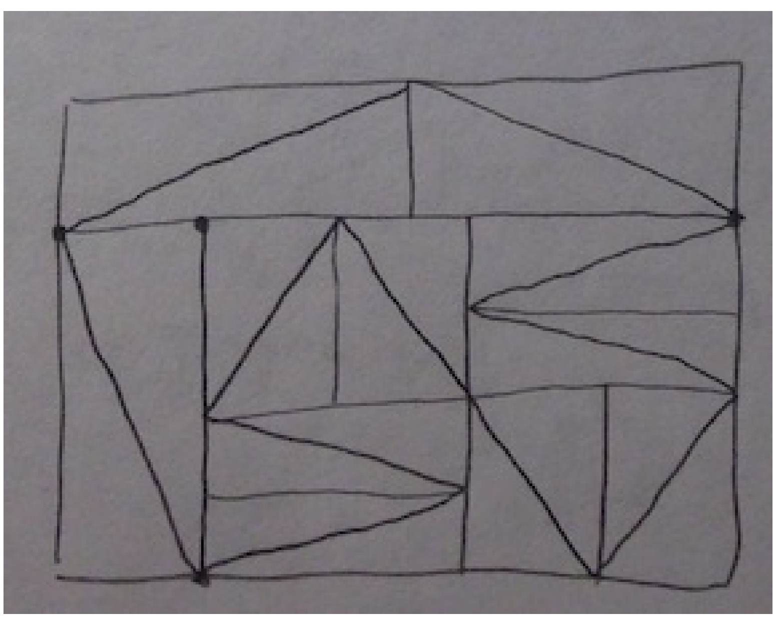
This one actually makes sense. There are 7 boys at the table.

9. The Big Island and The Small Island are both rectangular in shape and are divided into several rectangular counties. Each county has a road along one of the rectangle's diagonals. On each of the islands, these roads form a closed path which does not go twice through any of the points. Here is a map of the Small Island:



Draw a possible map of the Big Island if you know that it has an odd number of counties. How many counties does your island have?

Solution. Here is an example of an island with 11 counties:



10. Thirty three giants are guarding a cave. The Wicked Witch agreed to pay them 240 gold coins under the following conditions:

- The Wicked Witch divides the giants into several troops and pays each of the troops separately;
- Within each of the troops, the coins are divided equally between the giants, and the remainder is given back to the Wicked Witch.

(a) What is the biggest number of coins that the Wicked Witch can guarantee to herself if she can give the troops different numbers of coins (Note: the total number of coins given still must be 240)?

(b) What if she has to give each troop the same number of coins (independently of how many people are in each of the troops)?

Solution. Both solutions are based on the following inequality:

If the 33 giants are split up into troops $T_1, T_2, \ldots, T_k$ of sizes $n_1, n_2, \ldots, n_k$ respectively (so that $n_1 + n_2 + \cdots + n_k = 33$), then each troop T_i will have to give back *no more than* $n_i - 1$ gold coins (remainder is always less than divisor). Adding up all of these numbers, we see that the Witch can get no more than

$$(n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1) = (n_1 + n_2 + \cdots + n_k) - k = 33 - k$$

coins back from the giants. Thus, she would be best advised to split the giants up into as few troops as possible.

(a) If she splits the giants up into only one troop, it must have size 33, and she must give it 240. 240 has remainder 9 when divided by 33, so the Witch only gets 9 coins back if she decides to divide the giants into only one troop. However, if she divides the giants up into two troops, one with only 1 giant and the other with 32, and she gives the first troop 209 coins and the second troop 31, she gets 31 coins back. This is the best she can possibly do with $k \geq 2$, and since 31 is more than 9, this is the best return she can possibly guarantee.

(b) Again, if she splits the giants up into only one troop, she gets 9 coins back. Next, by making a table consisting of the remainders of 120 when divided by $1, 2, 3, \ldots, 33$, one sees that the best return the Witch can get if she splits the giants up into two troops is 27. If she splits the giants into three troops with sizes $3, 3, 27$, though, she gets 30 coins back. This is the best she can possibly do with $k \geq 3$ and since it's better than the 9 or 27 coins she can get back with $k = 1$ or 2, this is the best return she can possibly guarantee.

11. Put signs of mathematical operations and parentheses in such a way that you get a true statement:

$$\frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad = \quad 2$$

Solution. $(\frac{1}{4} \div \frac{1}{4}) + (\frac{1}{4} \div \frac{1}{4}) = 2.$

12. The following problem is attributed to Sir Isaac Newton: 70 cows eat the grass on a field in 24 days. 60 cows eat the grass on the same field in 30 days. How many cows would it take to eat all the grass in 96 days? (Hint: the grass continues to grow at a constant rate while the cows are eating it).

Solution. Let E be the rate at which a single cow eats grass (*without* the grass growing). Let G be the growth rate of the grass. Also, express E and G in the units: $\frac{\text{Fields of Grass}}{\text{Days}}$. Then the problem gives us the equations

$$(70E - G) \cdot 24 = 1$$
$$(60E - G) \cdot 30 = 1$$

We can solve this system of equations for E and G. We get $E = \frac{1}{1200}$ and $G = \frac{1}{60}$. Now to find the number of cows it would take to eat all the grass in 96 days we have to solve the equation $(nE - G) \cdot 96 = 1$. That is, the equation $(\frac{n}{1200} - \frac{1}{60}) \cdot 96 = 1$. The result is $n = 32.5$, so we need 33 cows to eat all the grass in 96 days.