

Math Circle  
Beginners Group  
October 11, 2015  
Solutions

**Warm-up problems**

1. Let us practice some exponents and powers! Calculate the following:

(a)  $2^2$   
 $= 4$

(b)  $2^2 \times 2^3$   
 $= 2^5 = 32$

(c)  $\frac{2^4}{2^2}$   
 $= 2^2 = 4$

(d)  $\frac{2^{2^2}}{4^4}$   
 $= \frac{2^2}{4^4} = \frac{2^4}{2^{2^4}} = \frac{2^4}{2^8} = \frac{1}{2^4} = \frac{1}{16}$

(e)  $(x \cdot y)^3$   
 $= x^3 y^3$

(f)  $(x^2 \cdot y)^2$   
 $= x^4 y^2$

2. Simplify the following expressions.

(a)  $\frac{3^{20} \cdot 7^6}{3^2 \cdot 7^2 \cdot 5^2 \cdot 2^2}$   
 $= \frac{30^{18} \cdot 7^4}{5^2 \cdot 2^2}$

(b)  $\frac{u^4 \cdot v^3}{(u^2 \cdot v)^4}$   
 $= \frac{1}{u^4 \cdot v}$

3. Which is bigger:  $2^{3^4}$  or  $4^{3^2}$ ?

$$2^{3^4} = 2^{12}$$

$$4^{3^2} = (2^2)^{3^2} = (2^2)^6 = 2^{12}$$

Therefore, the two are equal.

4. Find the value of  $x$  in  $\frac{4}{5} = \frac{32}{x}$ . Use cross-multiplication.

Using cross-multiplication,  $4x = 32 \times 5$ .

Therefore,  $x = 160 \div 40 = 40$ .

# Fractions and Decimals

October 11, 2015

## Solutions

1. Find the least common multiples for the following sets of numbers.

(a) 2 and 3  
LCM= 6

(b) 4 and 16  
LCM= 16

(c) 11 and 12  
LCM= 132

(d)  $n$  and  $n + 1$   
LCM=  $n(n + 1)$

(e) 5, 7 and 11  
LCM= 385

(f)  $n, n^2, n^3$   
LCM=  $n^3$

2. One of your friends who really likes fractions has offered you a series of trades. For each of the following, decide whether or not you should take the trade (without converting the fractions to decimals). All of the fruits of the same type are identical.

(a) He will give you  $\frac{3}{10}$  of his apple if you give him  $\frac{5}{14}$  of yours.

Since the fruits are identical, we should check which fraction is bigger.

The LCM of 10 and 14 is 70. Therefore,  $\frac{3}{10} = \frac{21}{70}$  and  $\frac{5}{14} = \frac{25}{70}$ .  $\frac{21}{70} < \frac{25}{70}$ .  
Therefore, you should not make the trade.

(b) He will give you  $\frac{9}{12}$  of his watermelon if you give him either  $\frac{11}{15}$  or  $\frac{15}{20}$  of yours.

The LCM of 12 and 15 is 60, and the LCM of 12 and 20 is also 60. Therefore,  
 $\frac{9}{12} = \frac{45}{60}$ ,  $\frac{11}{15} = \frac{44}{60}$  and  $\frac{15}{20} = \frac{45}{60}$ . Therefore, it is irrelevant whether or not  
you pick  $\frac{15}{20}$ . So you should trade  $\frac{11}{15}$  of your watermelon for  $\frac{9}{12}$  of his.

(c) He will give you  $\frac{43}{44}$  of his mango if you give him  $\frac{43}{3}$  of  $\frac{1}{15}$  of yours.

$\frac{43}{3}$  of  $\frac{1}{15}$  equals  $\frac{43}{45}$ .  $\frac{43}{44} > \frac{43}{45}$ , so you should make the trade.

3. Find two fractions between  $\frac{2}{3}$  and  $\frac{4}{5}$ .

A fraction that is halfway between  $\frac{2}{3}$  and  $\frac{4}{5}$  can be found by taking the average of  $\frac{2}{3}$  and  $\frac{4}{5}$ .

$$\frac{2}{3} + \frac{4}{5} = \frac{10+12}{15} = \frac{22}{15}$$

$\frac{22}{15} \div 2 = \frac{11}{15}$  We can obtain another fraction by finding the average of  $\frac{2}{3}$  and

$$\frac{11}{15} \text{ (or in fact, also the average of } \frac{4}{5} \text{ and } \frac{11}{15}\text{). } \frac{2}{3} + \frac{11}{15} = \frac{10+11}{15} = \frac{21}{15} = \frac{7}{5}$$

$\frac{7}{5} \div 2 = \frac{7}{10}$  Therefore, the two fractions between  $\frac{2}{3}$  and  $\frac{4}{5}$  are  $\frac{7}{10}$  and  $\frac{11}{15}$ .

4. Convert the following fractions into decimals and say which one is bigger.

(a)  $\frac{7}{4}$ ,  $\frac{42}{25}$

One can use long division.

$$\frac{7}{4} = 1.75, \frac{42}{25} = 1.68$$

$$\frac{7}{4} > \frac{42}{25}$$

(b)  $\frac{3333}{500}$ ,  $\frac{667}{100}$

$$\frac{3333}{500} = 6.666, \frac{667}{100} = 6.67$$

$$\frac{3333}{500} < \frac{667}{100}$$

(c)  $\frac{9}{4}$ ,  $\frac{226}{1000}$

$$\frac{9}{4} = 2.25, \frac{226}{1000} = 0.226$$

$$\frac{9}{4} > \frac{226}{1000}$$

How else can you compare fractions without converting them to decimals or taking a common denominator?

We can compare fractions using cross-multiplication.

If we have  $\frac{a}{b}$  and  $\frac{d}{c}$ , and  $ac > bd$ , then  $\frac{a}{b} > \frac{d}{c}$ , or vice-versa. This is true only if both fractions are positive or both are negative.

5. Convert the following fractions into decimals.

(a)  $\frac{1}{3}$   
= 0.3333...

(b)  $\frac{11}{13}$   
= 0.846153846153...

$$(c) \frac{1}{7} \\ = 0.1428571428571\dots$$

$$(d) \frac{25}{99} \\ = 0.252525\dots$$

6. What do you think is common in the decimals expansions in problem 4?

All decimal expansions are finite, which means that they terminate or do not go on forever.

7. What do you think is common in the decimals expansions in problem 5?

All decimal expansions go on forever and repeat themselves.

8. What do you think might be the cause of this? (Think about the denominators.)

(a) Find the prime factorization of the following numbers:

$$i. 4 \\ = 2 \times 2 = 2^2$$

$$ii. 25 \\ = 5 \times 5 = 5^2$$

$$iii. 100 \\ = 2 \times 2 \times 5 \times 5 = 2^2 \times 5^2$$

$$iv. 500 \\ = 2 \times 2 \times 5 \times 5 \times 5 = 2^2 \times 5^3$$

$$v. 1000 \\ = 2 \times 2 \times 2 \times 5 \times 5 \times 5 = 2^3 \times 5^3$$

(b) Find the prime factorization of the following numbers:

$$i. 3 \\ = 3$$

$$ii. 7 \\ = 7$$

$$iii. 13 \\ = 13$$

$$iv. 99 \\ = 3 \times 3 \times 11 = 3^2 \times 11$$

(c) Can you now say what is the difference between the denominators of problem 4 and problem 5?

The denominators in problem 4 have prime factors of only 2 and 5, while the denominators in problem 5 have prime factors other than 2 and 5.

9. To summarize, fractions with denominators that have prime factors of only 2 and 5 have terminating decimal expansions. Fractions whose denominators have prime factors other than 2 and 5 have decimal expansions that go on forever (non-terminating decimals).
10. Can you give a name to this property of fractions? Be creative!

Students can come up with anything that describes the property.

11. Circle all the fractions that have terminating decimal expansions. Underline all the fractions that have non-terminating decimal expansions.

Fractions with terminating decimal expansions:  $\frac{3}{5}$ ,  $\frac{64}{125}$ ,  $\frac{19}{256}$ ,  $\frac{11}{160}$ ,  $\frac{7}{35}$ ,  $\frac{57}{6}$

Fractions with non-terminating decimal expansions:  $\frac{1}{7}$ ,  $\frac{13}{30}$ ,  $\frac{1}{300}$ ,  $\frac{37}{1200}$

12. Can you think of a fraction whose decimal representation neither terminates nor repeats itself?

Nope!

13. Convert the following decimals into fractions. What do you already know about the fractions below?

Since all the decimals below are terminating, the denominators of the fractions would have prime factors of only 2 and 5.

$$(a) \ 0.80 \\ = \frac{80}{100} = \frac{4}{5}$$

$$(b) \ 0.105 \\ = \frac{105}{1000} = \frac{21}{200}$$

$$(c) \ 0.25 \\ = \frac{25}{100} = \frac{1}{4}$$

$$(d) \ 0.125 \\ = \frac{125}{1000} = \frac{25}{200} = \frac{1}{8}$$

Let us prove the property that we discovered above: Fractions whose denominators only have powers of 2 and 5 in their decomposition have terminating decimal expansions.

- (a) Prove that if a fraction has denominator  $2^n 5^n$ , i.e. it can be written in the form  $\frac{a}{2^n 5^n}$ , then the decimal equivalent terminates.

$$2^n 5^n = 10^n$$

$$\frac{a}{2^n 5^n} = \frac{a}{10^n}$$

We know that  $\frac{1}{10^n}$  is equal to 0.00.....01 with  $n - 1$  zeroes. When we divide a number by 10, the decimal only shifts by one place (since we are using the number system of base 10). Therefore, dividing a number by  $10^n$  will shift the decimal by  $n - 1$  places, and it will yield a terminating decimal expansion.

- (b) Prove that if a fraction has denominator  $2^n \cdot 5^{n+k}$ , i.e. it can be written in the form  $\frac{a}{2^n \cdot 5^{n+k}}$ , then the decimal equivalent terminates. (Hint: Multiply the numerator and denominator by an appropriate number so that you can use part (a).)

To be able to use part (a), we need the denominator of  $\frac{a}{2^n \cdot 5^{n+k}}$  to be equal to 10 or a power of 10.

$$\frac{a}{2^n \cdot 5^{n+k}} \times \frac{2^k}{2^k}$$

$$= \frac{2^k \cdot a}{2^{n+k} \cdot 5^{n+k}} = \frac{2^k \cdot a}{10^{n+k}}$$

We can now use part (a). Since the denominator is a power of 10, we know that the decimal equivalent is terminating.

- (c) Similarly, prove that if a fraction has denominator  $2^{n+k} \cdot 5^n$ , then the decimal equivalent terminates.

Like in part (b), we need the denominator of  $\frac{a}{2^{n+k} \cdot 5^n}$  to be equal to 10 or a power of 10.

$$\frac{a}{2^{n+k} \cdot 5^n} \times \frac{5^k}{5^k}$$

$$= \frac{5^k \cdot a}{2^{n+k} \cdot 5^{n+k}} = \frac{5^k \cdot a}{10^{n+k}}$$

Since the denominator is a power of 10, we know that the decimal equivalent is terminating.

- (d) State the property that denominators of fractions with terminating decimals must have.

If a fraction has a denominator with prime factors of only 2 and 5, then it has a denominator that looks like one in the previous three parts of this problem. We have seen that the decimal expansions of all these fractions terminate.

The property states that if the denominator of a fraction has prime factors of only 2 and 5, then the decimal expansion of the fraction must terminate.

14. Can you prove that fractions whose denominators have prime factors other than 2 and 5 do not have terminating decimal expansions?

(a) What is an example of such a fraction?

$\frac{1}{3}$ , or any fraction of the lowest form  $\frac{p}{q}$ , such that  $q$  has prime factors other than 2 or 5.

(b) Let us assume that this fraction does have a terminating decimal expansion. What form can fractions with terminating decimal expansions be written in?

We know from problem 13 that such fractions have the form  $\frac{a}{10^n}$ . Therefore, if  $\frac{p}{q}$  does have a terminating decimal expansion,  $\frac{p}{q} = \frac{a}{10^n}$ .

(c) Explain why the fractions you obtained above can never be equal using cross-multiplication.

Using cross-multiplication,  $10^n \cdot p = a \cdot q$ .

However, this is never possible because  $q$  and  $10^n$  do not have any common factors, and  $p$  is not divisible by  $q$ .

15. Other than using long division, can you think of another way to convert fractions that have the property you discovered in problem 13 (d) into decimals? (Hint: Multiply the numerator and denominator by an appropriate number.) Explain this method in your words below.

We should multiply the numerator and denominator by an appropriate number such that denominator is equal to  $10^n$ . Then we can shift the decimal place by  $n - 1$  times.

16. Use the method you found above to convert the following fractions into decimals.

$$(a) \frac{3}{4} = \frac{3}{4} \times \frac{25}{25} = \frac{75}{100} = 0.75$$

$$(b) \frac{18}{20} = \frac{18}{20} \times \frac{5}{5} = \frac{90}{100} = 0.9$$

$$(c) \frac{137}{250} = \frac{137}{250} \times \frac{4}{4} = \frac{548}{1000} = 0.548$$

$$(d) \frac{11}{50} = \frac{11}{50} \times \frac{2}{2} = \frac{22}{100} = 0.22$$

$$(e) \frac{5}{16} = \frac{5}{16} \times \frac{625}{625} = \frac{3125}{10000} = 0.3125$$



$$\begin{aligned}
 \text{(f)} \quad & \frac{2^{2^2}}{5^{2^2}} \\
 &= \frac{2^4}{5^4} = \frac{16}{625} = \frac{16}{625} \times \frac{16}{16} = \frac{256}{10000} = 0.0256
 \end{aligned}$$

17. Prove that the decimal equivalents of the following fractions terminate.

$$\begin{aligned}
 \text{(a)} \quad & \frac{3^{20} \cdot 7^6}{3^2 \cdot 7^2 \cdot 5^2 \cdot 2^2} \\
 &= \frac{3^{18} \cdot 7^2}{5^2 \cdot 2^2}
 \end{aligned}$$

Since the denominator of the fraction has prime factors of only 2 and 5, its decimal equivalent terminates.

$$\begin{aligned}
 \text{(b)} \quad & \frac{2013 \cdot 2012}{503 \cdot 11 \cdot 61 \cdot 3 \cdot 5^4} \\
 &= \frac{2013 \cdot 4}{11 \cdot 61 \cdot 3 \cdot 5^4} \\
 &= \frac{183 \cdot 4}{61 \cdot 3 \cdot 5^4} \\
 &= \frac{3 \cdot 4}{3 \cdot 5^4} \\
 &= \frac{4}{5^4}
 \end{aligned}$$

Since the denominator of the fraction has prime factors of only 2 and 5, its decimal equivalent terminates.