

# VOTING THEORY

## 1 Single-winner voting systems

### 1.1 A warm-up exercise

Hello, and welcome to the Math Circle elections! We've got a great year of math coming up, but before we can get to any of that, we need to decide who will be in charge of our group. The problem is, while we know who the candidates are (A, B, C, D, and E), we don't know how the election should work. The latest poll gave the following preferences:

- 31: A > D > C > E > B
- 20: B > D > C > E > A
- 19: D > C > E > B > A
- 16: E > C > B > A > D
- 14: C > E > D > B > A

This means that there are 31 voters who have *A* as their top choice, *D* as their second choice, then *C*, then *E*, and finally *B* as their least favourite candidate. And so on.

Come up with 5 voting methods which all sound fair, and such that each one of them will make a different candidate win.

Don't read ahead yet if you do not want to be spoiled!

## 1.2 Voting systems and voting criteria - the short list

- COMMON VOTING SYSTEMS
  - **Plurality:** Each voter votes for their preferred candidate. The candidate with the most votes wins.
  - **Top-two run-off:** First, each voter votes for their preferred candidate. We select the two candidates with the most votes, and we have a second round only with them. The candidate among the two with the most votes wins.
  - **Borda Count:** If there are  $N$  candidates, each voter assigns  $N - 1$  points to their preferred candidate,  $N - 2$  votes to their second preferred candidate, and so on, to 0 points for their least-preferred candidate. The candidate with the most points wins.
  - **Instant Run-off Voting:** Each voter votes for their favourite candidate. The candidate with the least votes is eliminated. Repeat until there is only one candidate left.
  - **Survivor:** Each voter votes for their least favourite candidate. The candidate with the most votes is eliminated. Repeat until there is only one candidate left.
- SIMPLE VOTING CRITERIA
  - **Majority criterion:** If a majority (i.e. more than 50%) of the voters have  $X$  as top choice, then  $X$  wins.
  - **Majority-loser criterion:** If a majority of voters have  $X$  as their last choice, then  $X$  does not win.
  - **Pareto Criterion:** If every voter prefers  $X$  to  $Y$ , then  $Y$  does not win.
  - **Condorcet criterion:** Assume there is a candidate  $X$  such that, for any other candidate  $Y$ ,  $X$  would win against  $Y$  in a head-to-head race just between the two of them. Then  $X$  wins.

**Exercise:** For each of the five voting systems above, which ones of the four criteria do they pass or do they fail? For example, Plurality passes majority criterion. On the other hand, Survivor fails the Majority Criterion because we can construct a scenario in which a candidate with more than 50% of the votes loses the election.