

Problems from Moscow Math Olympiads

LA Math Circle
High School II

October 4, 2015

1. Find all ways to place four soccer players in a field so that the pairwise distances between them are 1, 2, 3, 4, 5, and 6 meters, or prove that no such placement is possible.

2. Solve the equation

$$(x + 1)^{63} + (x + 1)^{62}(x - 1) + (x + 1)^{61}(x - 1)^2 + \cdots + (x - 1)^{63} = 0.$$

3. An arithmetic progression consists of integers. The sum of the first n terms of this progression is a power of two. Prove that n is also a power of two.

4. Two points, A and B , are marked on the graph of the function $y = 1/x$, $x > 0$. Denote by H_A and H_B the feet of the perpendiculars dropped from these points to the x axis and by O the origin. Prove that the area of the figure bounded by the lines OA and OB and the arc AB of the graph equals the area of the figure bounded by the lines AH_A and BH_B , the x axis, and the arc AB .

5. Prove that any quadratic polynomial can be represented as the sum of two quadratic polynomials each having discriminant zero.

6. Is it possible to find five consecutive terms of the sequence $a_n = 1 + 2^n + \dots + 5^n$ each of which is divisible by 2005?

7. There are 15 cities in a country. Some of them are joined by air routes, which belong to three companies. It is known that even if one of the companies discontinues its flights, it will still be possible to get from any city to any other (possibly with transfers), using flights of the other two companies. What is the smallest number of air routes in this country?

Hint: If you think the answer is n , then you have to do two things: (a) Prove that it's impossible to have less than n routes. (b) Give an example that shows that n is possible.

8. On the graph of a polynomial with integer coefficients, two points with integer coordinates are chosen. Prove that if the distance between them is an integer, then the segment joining them is parallel to the x -axis.

9. The tangents of a triangle's angles are positive integers. What are the possible values for these tangents?

10. In an acute triangle ABC , draw the altitudes AH_A , BH_B , and CH_C . Then consider the orthocenter (intersection point of the altitudes) of each of the triangles AH_BH_C , BH_AH_C , and CH_AH_B . Show that these three points form a triangle congruent to $H_AH_BH_C$.