

# Combinatorics (Part I)

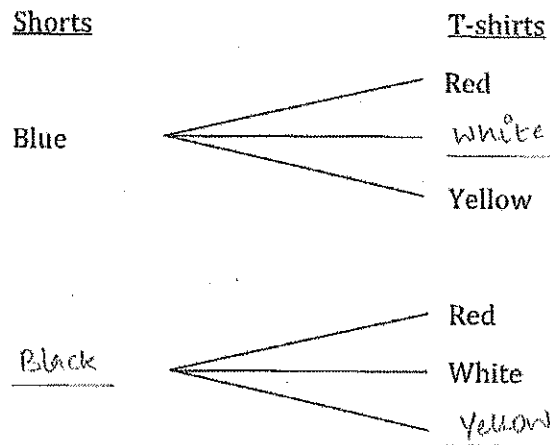
BEGINNERS 02/01/2015

*Combinatorics is the study of the methods of counting.*

## Part I: Multiplication Principle

1. If you have two pairs of shorts, one blue and one black, and three T-shirts, red, white and yellow, how many outfits can you make?

Let's see how many different *combinations* of T-shirts and shorts we can have. Fill in the blanks.



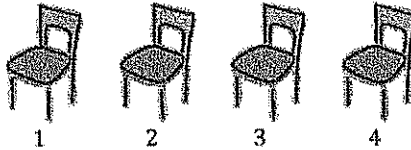
We see that we can have

1. Blue shorts and Red T-shirt
2. Blue shorts and white T-shirt
3. Blue shorts and Yellow T-shirt
4. Black shorts and Red T-shirt
5. Black shorts and white T-shirt
6. Black shorts and yellow T-shirt

This is 6 outfits, or 6 *combinations*, in all.

But sometimes, we can't list out all the possible ways. Let us look at a similar problem.

2. If there are four chairs and four students – Ashley, Benjamin, Caitlyn and Daniel – in a classroom, in how many ways can the students be seated on the chairs?



- i. How many different students have the option of sitting on the first chair? 4
- ii. If one of them sits on the first chair, how many can sit on the second? 3
- iii. Now, how many can sit on the third chair? 2
- iv. How many can sit on the fourth chair? 1

So, let's see how many total ways of arranging the students on the chairs there are.

For each choice of the student who sits on the first chair (4 in total), we can then consider all the choices of who can sit on the second (3 in total).

This gives us  $4 \times 3 = 12$  different arrangements for the first two chairs.

Then, for all the possible choices of who can sit on the first and the second chairs, we can consider all the choices of who can sit on the third chair (2 in total).

So, we can have  $4 \times 3 \times 2 = 24$  arrangements for the first three chairs.

Then, for the different possible arrangements of students sitting on the first three chairs, we can consider how many students can sit on the last chair.

This gives us  $4 \times 3 \times 2 \times 1 = 24$  total arrangements for the four chairs.

So, there are 24 ways in which the all students can be seated on the four chairs.

It looks like we just multiplied the different number of ways to fill each individual other.

Now go back to the first question and see what we did there. What can we multiply together to get the number of all possible outfits?

$$\begin{aligned} & \text{no. of colors of shorts} \times \text{no. of colors of T-shirts} \\ & 2 \times 3 \\ & = 6 \end{aligned}$$

We can generalize:

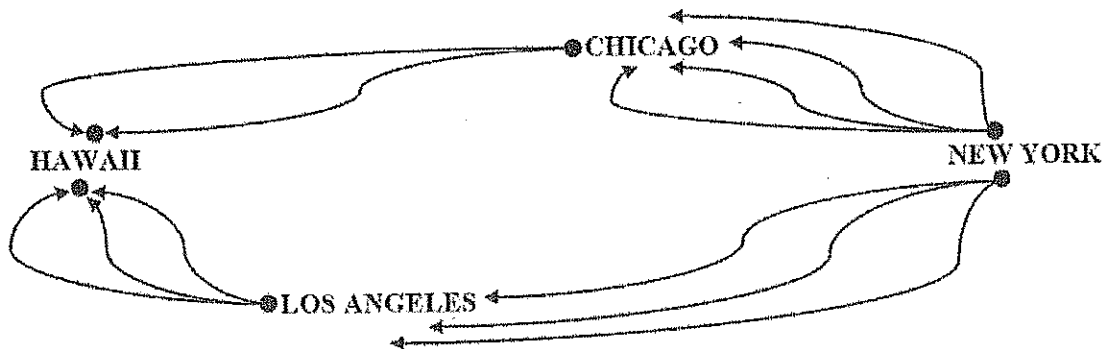
If a process can be broken down into a number of steps, and all the steps are independent of each other like we saw above (choosing a particular pair of shorts does not affect your choice of T-shirt), then the total number of outcomes of the process is the number of possibilities in the first step times the number of possibilities in the second step times the number of possibilities in each step until the last one. This is called the *Multiplication Principle* of combinatorics.

## Part II: Addition Principle

Let us look at a new idea now.

Max wants to go from New York to Hawaii. He has two options. One, he can go from New York to LA and then to Hawaii, like we did above. Or, he can go from New York to Chicago, where his grandmother lives, and then to Hawaii. There are four ways to get from New York to Chicago: by train, by car, by teleportation or by plane. And there are two ways to get from Chicago to Hawaii: by plane and by teleportation. In how many ways can Max get to Hawaii from New York?

We can break this down as follows:



How many ways are there to go from New York to Hawaii via Chicago?

$$4 \times 2 = 8$$

How many ways are there to go from New York to Hawaii via LA?

$$3 \times 3 = 9$$

How many ways are there to go from New York to Hawaii in all?

$$8 + 9 = 17$$

It is easy to see that you add the number of ways in both travel plans to get the total number of ways. This is the *Addition Principle* of combinatorics: When you have a choice of methods for performing a procedure (like, via Chicago or via Los Angeles), then the number of ways of performing the procedure is found by adding the number of ways for all the methods individually.

Practice Problems

1. There are four fiction books and seven science books in a bookstore. In how many ways can you buy a fiction and a science book?

$$4 \times 7 \\ = 28$$

: 4 ways to buy a fiction book  
x 7 ways to buy a science book

2. A sweater has to be knitted with three different colors of yarn: one for the left sleeve, one for the right sleeve and one for the body. If we have nine differently colored yarns available, how many different kinds of sweaters can be made?

$$9 \times 8 \times 7 \\ = 72 \times 7 \\ = 504$$

3. A store carries 7 styles of pants. For each style, there are 10 different waist sizes, 8 different length options and 2 color choices. How many different pairs of pants does the store sell?

$$7 \times 10 \times 8 \times 2 \\ = 70 \times 16 \\ = 1120$$

4. There are three ways of going from New York to Los Angeles: by train, by plane and by teleportation. And there are three ways of going from Los Angeles to Hawaii: by plane, by teleportation and by sea. How many ways are there to go from New York to Hawaii via Los Angeles?

$$3 \times 3 = 9$$

### Part III: Multiple Independent Events

A natural number is odd-looking if all its digits are odd. How many odd-looking six-digit numbers are there?



- |      |   |          |
|------|---|----------|
| i.   | How many different odd digits are there?                | <u>5</u> |
| ii.  | How many different digits can appear in the first box?  | <u>5</u> |
| iii. | How many different digits can appear in the second box? | <u>5</u> |
| iv.  | How many different digits can appear in the third box?  | <u>5</u> |
| v.   | How many different digits can appear in the fourth box? | <u>5</u> |
| vi.  | How many different digits can appear in the fifth box?  | <u>5</u> |
| vii. | How many different digits can appear in the sixth box?  | <u>5</u> |

How many odd-looking six-digit numbers are possible? Write the answer as a product.

$$5 \times 5 \times 5 \times 5 \times 5 \times 5$$

Which principle of combinatorics did you use?

Multiplication

Let's see if there is another way to write the answer.

$5 \times 5 \times 5 \times 5 \times 5 \times 5$  can also be written as  $5^6$ , which is read as "5 to the 6<sup>th</sup> power." It means that we are multiplying 5 by itself six times.

Therefore, there are  $5^6 = 15,625$  odd-looking six-digit numbers possible. That's a big number!

### Practice Problems

1. Angie has to draw one card from a deck with 52 playing cards. In how many ways can she choose a king or a queen?

4 ways to choose a king

4 ways to choose a queen

$$\Rightarrow 4 + 4 = 8 \text{ ways to choose a king or a queen.}$$

2. If a bookstore sells five fiction books, four science books and three children's books, in how many ways can you buy two books of different kinds?

Two books of different kinds can be:

fiction & science  
or science & children's  
or fiction & children's

$$\text{So, } 5 \times 4 + 4 \times 3 + 3 \times 5 \\ = 20 + 12 + 15 = 47$$

} Multiplication  
and addition  
principles

3. A college acting troupe has 6 junior women, 8 junior men, 5 senior women and 4 senior men. In how many ways can the teacher select a senior couple or a junior couple to play the roles of Romeo and Juliet?

$$\text{Ways to select a senior couple: } 5 \times 4 = 20$$

$$\text{Ways to select a junior couple: } 6 \times 8 = 48$$

$$\text{Ways to select a senior OR junior couple} = 20 + 48 \\ = 68$$

4. In a movie rental store, there are five different English movies, three French movies, two Russian movies and four Hindi movies. How many ways are there to rent three movies in different languages?

Three movies of different languages can be:

English, French, Russian  
OR English, Russian, Hindi  
OR Hindi, French, Russian  
OR English, Hindi, French

$$\text{So, } 5 \times 3 \times 2 + 5 \times 2 \times 4 + 4 \times 3 \times 2 + 5 \times 4 \times 3 \\ = 30 + 40 + 24 + 60 \\ = 154$$

*Permutations* are arrangements that can be made by placing different objects in a row. Like we did for the question of students on the chairs in the beginning, the order in which you place these objects is important.

1. How many three-digit numbers can be made using the digits 2, 4 and 6 only once each?

In the case of numbers, the order of how the digits are arranged matters. Of course, the number 24 is different from 42; 246 is different from 264 and 426, and so on.



- i. In how many ways can you fill the hundred's place box? 3  
ii. In how many ways can you fill the ten's place box? 2  
iii. In how many ways can you fill the one's place box? 1

How many permutations are possible in all?

$$3 \times 2 \times 1$$

Which principle of combinatorics did you use?

Multiplication

2. In how many ways can we award gold, silver and bronze medals among eight contestants?

The order matters. One of the *permutations* "gold, bronze, silver" is different from the other permutation of "gold, silver, bronze."

- i. How many students can you give the gold medal to? 8  
ii. How many students are left to receive the silver medal? 7  
iii. How many students are left to receive the bronze medal? 6

How many permutations are possible in all?

$$8 \times 7 \times 6$$

Which principle of combinatorics did you use?

Multiplication

Practice Problems

1. If you toss a coin, how many different outcomes can you get?

Heads, tails.  
 $\Rightarrow$  2 outcomes

2. If you now toss two coins - a nickel and a dime - how many different outcomes can you get? Remember that a heads on the nickel is different from a heads on the dime.

4 outcomes = Heads on nickel, Tails on nickel, Heads on dime, Tails on dime



① outcome on coin 1.  
 ② outcome on coin 2.

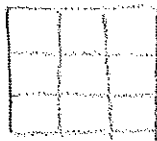
So,  $2 \times 2 = 4$  ways

3. If you then toss three coins, a penny, a nickel and a dime, how many different outcomes can you get?



$$2 \times 2 \times 2 = 2^3 \\ = 8 \text{ ways}$$

4. Each box in a  $3 \times 3$  square can be colored using one out of three colors. How many different colorings of the grid are there?



$$3 \times 3 \times \dots \times 3 = 3^9 \\ \underbrace{\hspace{1.5cm}}_{9 \text{ times}}$$

5. The alphabet of a tribal language contains 4 letters. A word in this language can be formed by any combination of these letters and the words can have between 1 and 5 letters. How many words are there in this language? You may leave your answer in the form of powers. (Hint: Calculate separately the number of one-letter, two-letter, three-letter, four-letter and five-letter words in this language.)

one-letter  $\square$   $4$   
 two-letter  $\square \square$   $4 \times 4 = 4^2$   
 three-letter  $\square \square \square$   $4 \times 4 \times 4 = 4^3$   
 four-letter  $\square \square \square \square$   $4 \times 4 \times 4 \times 4 = 4^4$   
 five-letter  $\square \square \square \square \square$   $4 \times 4 \times 4 \times 4 = 4^5$

$$= 4 + 4^2 + 4^3 + 4^4 + 4^5 \text{ words}$$



What is different between example 1 and example 2?

In Example 1, we used up all the available objects (digits 2, 4 and 6) for arrangement.

In Example 2, we did not arrange the rest of the

How many four-digit numbers can you make using the digits 2, 4, 6 and 8?  $\underline{4} \times \underline{3} \times \underline{2} \times \underline{1}$  <sup>5 students</sup>

How many five-digit numbers can you make using the digits <sup>1</sup>0, 2, 4, 6 and 8?  $\underline{5} \times \underline{4} \times \underline{3} \times \underline{2} \times \underline{1}$

Both of these involve the product of decreasing whole numbers. These are known as factorials, written in math as  $n!$

It is generally helpful to write these out in descending order.

$$8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

$$6! = \underline{6} \times \underline{5} \times \underline{4} \times \underline{3} \times \underline{2} \times \underline{1}$$

$$5! = \underline{5} \times \underline{4} \times \underline{3} \times \underline{2} \times \underline{1}$$

### Practice Problems

1. How many permutations are there of the set {red, blue, yellow, green, purple, orange}?

$$6!$$

2. How many permutations are there of a set of  $n$  distinct elements?

$$n!$$

## Combinatorics (Part II)

BEGINNERS 02/08/2015

### Part IV: Permutations (Continued)

*Permutations* are arrangements that can be made by placing objects in a row. The order of the objects is important.

1. How many three digit numbers can you write using the digits 3, 3 and 4?

(a) Let's say that we have a 3, a **bold 3** and a 4. Write down all the numbers that you can make with the digits 3, 3 and 4.

3 3 4      3 4 **3**  
3 **3** 4      4 **3** 3  
3 4 3      4 3 **3**

(b) Considering the 3 and the bold 3 as two different digits, how many different numbers are there?

6

Even though they are colored differently, the 3 and the bold 3 have the same meaning:

334 is the same number as 334.

433 is the same number as 433.

343 is the same number as 343.

For every permutation that we write, there will be another one in which the positions of the 3s are switched. And both numbers will obviously be the same!

Therefore, there are some *repetitions* among the numbers above.

(c) Taking into account that 3 and bold 3 mean the same digit, how many different numbers written with digits 3, 3 and 4 are there?

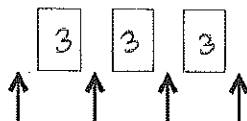
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Here is how we computed the number of arrangements of the digits 3, 3, 4:

First, pretend that the two 3s are different (one of them is bold). Then the number of arrangements is equal to  $3! = 6$ . Then, take into account that the arrangements which are obtained from each other by switching the two 3s are actually the same. As a result, we get  $3! / 2! = 3$ .

2. How many four-digit numbers can you write using the digits 3, 3, 3, and 4?

(a) Notice that three of the digits are equal to 3. This means that we just have to decide where to put the digit 4.



This can be done in 4 ways.

Thus, the total number of permutations is 4.

(b) Now we will compute the number of rearrangements using the same method as in the first problem. Let's say that we have a 3, a bold 3, an underlined 3, and a 4. In how many ways can you rearrange 3, **3**, 3, and 4?

$$4! = 24$$

(c) Every two rearrangements which differ just by the order of 3s are the same. In how many ways can you rearrange 3, **3**, 3 in three slots?

$$3!$$

(d) How many distinct four-digit numbers written with 3, 3, 3, and 4 are there? Does your answer agree with the result of part (a)?

$$\frac{4!}{3!} = 4$$

Yes

Practice Problems

5. There are 39 cities in a country, each with one airport. Every pair of them is connected by an air route. How many air routes are there?

$$\frac{39 \times 38}{2}$$

6. Josh's mother has two apples, three bananas, and four oranges. Every morning, for nine days, she gives one fruit to her son for breakfast. How many ways are there to do this?

You can think of this problem as arranging the 9 letters A, A, B, B, B, O, O, O, O in a row.

$$\frac{9!}{2! 3! 4!}$$

7. How many "words" can be written using exactly five As and at most three Bs?

$$\text{Exactly 5 As and no B : } \frac{5!}{5!} = 1$$

$$\text{and 1 B : } \frac{6!}{5!} = 6$$

$$\text{and 2 Bs : } \frac{7!}{5! 2!}$$

$$\text{and 3 Bs : } \frac{8!}{5! 3!}$$

So, total no of words =

$$1 + 6 + \frac{7!}{5! 2!} + \frac{8!}{3! 5!}$$

$$7 + 21 + 56 = 84$$

Word Bank

1. In how many ways can you rearrange the letters of the word IOWA?

$$4! = 24$$

2. In how many ways can you rearrange the letters of the word ARIZONA?

- (a) Pretending that the two letters A are different, how many permutations do you have?

$$7!$$

- (b) Now taking into account that both As are identical, how many permutations do you have?

$$\frac{7!}{2!}$$

3. In how many ways can you rearrange the letters of the word COLORADO?

$$\frac{8!}{3!}$$

*You can leave the answer as is.*

4. In how many ways can you rearrange the letters of word TENNESSEE?

$$\frac{9!}{2!2!4!}$$

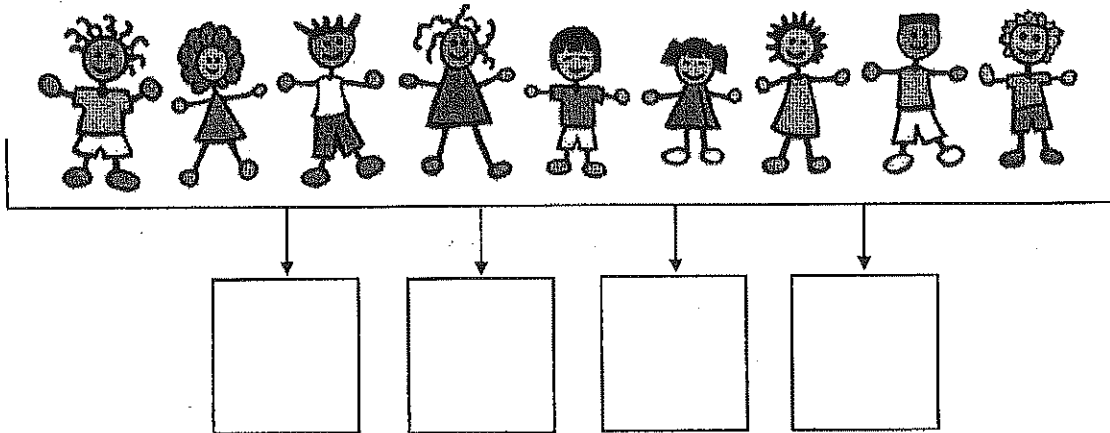
*Total no. of characters: 9*

*Repetitions: 2 Ns, 2 Ss, 4 Es*

Part V: Combinations

Sometimes, the order in which you arrange things doesn't matter. For example, Ms. Cranberry has to choose four students from her class of nine to send for a mathematical contest. Does the order in which she picks the first, second, third and fourth student matter?

No, it does not!



- i. How many different options does Mrs. Cranberry have for the first student? 9
- ii. After selecting the first, how many options does she have for the second? 8
- iii. Now, for the third? 7
- iv. For the fourth? 6

We see that there are  $9 \times 8 \times 7 \times 6$  ways of choosing the students. Let us call this number A. However, this gives us the answer for the number of permutations (i.e., the order matters!)

Forget about the order now. If the names of the four students picked are Abe, Gus, Rob and Zed, it doesn't matter if the order is "Gus, Zed, Abe, Rob" or "Zed, Abe, Rob, Gus." So, in how many ways can you arrange these four students amongst themselves?  $4 \times 3 \times 2 \times 1$   
Let us call this number B. B is simply the number of ways in which you can rearrange four students. It is also the number of repetitions, right?

Similarly to what we did above when we had repetitions or *redundancies*, let's divide A by B. We find the number of all the permutations and divide by the number of all the *redundancies*.

What does  $\frac{A}{B}$  show? Explain in your own words.

It shows the total no. of ways of choosing 4 children from a group of 9. We divide by  $4!$  because the order does not matter.

Practice Problems

1. There are four kittens at the pet store. Your mom says that you may choose two to take home. In how many different ways can you choose two kittens from the litter of four?

$$\frac{4 \times 3}{2!} = 6$$

2. The coach of the UCLA football team has to choose a captain and a deputy from his team of 10 students. How many ways are there to do that?

$$10 \times 9 = 90$$

(The order matters because captain and deputy are not equivalent.)

3. The coach of the USC football team, however, wants to choose a captain and a deputy as well as three assistants from his team of 10 students. In how many ways can he pick these four leaders?

$$\frac{10 \times 9 \times 8 \times 7 \times 6}{3!}$$

(because the order in which the assistants are picked does not matter)

4. Mr. Pi has to choose three girls and three boys to send for a debate. There are 14 girls and 11 boys in his class. In how many ways can he make the team?

$$\frac{14 \times 13 \times 12}{3!} \times \frac{11 \times 10 \times 9}{3!}$$

5. On any given night, there can be between zero and four babysitters at home in the Simpsons' house. The babysitter company has eight employees from which it can choose to send babysitters to the Simpsons. How many possible combinations of babysitters can be at the house?

$$\text{Zero babysitters} = 1 \text{ way}$$

$$1 \text{ babysitter} = 8 \text{ ways}$$

$$2 \text{ babysitters} = \frac{8 \times 7}{2!}$$

$$3 \text{ babysitters} = \frac{8 \times 7 \times 6}{3!}$$

$$4 \text{ babysitters} = \frac{8 \times 7 \times 6 \times 5}{4!}$$

Total no. of ways:

$$1 + 8 + \frac{8 \times 7}{2!} + \frac{8 \times 7 \times 6}{3!} + \frac{8 \times 7 \times 6 \times 5}{4!}$$

$$= 163$$