LAMC Intermediate I **May 10, 2015**

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Gravity and Energy

The following is known as the *Law of Universal Gravitation*. Two objects of the masses M and m attract each other with the force proportional to the masses and inversely proportional to the squared distance r between the objects.

$$F = \gamma \frac{Mm}{r^2} \tag{1}$$

If we take the mass of the Earth for M and the (average) radius of our planet for r, then the formula 1 describes the force our home planet exerts on a mass m on its surface.

$$F = mg, \quad \text{where} \quad g = \gamma \frac{M}{r^2} \tag{2}$$

The number $g = 9.8\ m/s^2$ is called the *free fall acceleration*.

Recall that acceleration is the rate of change of speed. If a body travelling at the original speed v_0 undergoes a constant acceleration a for the time period t, then the formula for the speed of the motion is

$$v(t) = v_0 + at. \tag{3}$$

If time is measured in seconds and the distance is measured in meters, then every second the speed $v(t)$ increases by a meters per second (or decreases in the case of the negative a).

Problem 1 *A body is released from rest not too far from the Earth's surface. Fill out the following table where t is time in seconds and $v(t)$ is the speed at which the body travels at the moment t.*

t	*0*	*1*	*2*	*3*
v				

If a body travelling along a straight line at the original speed v_0 undergoes a constant acceleration a for the time period t, then the distance it covers is

$$s(t) = s_0 + v_0 t + \frac{at^2}{2}. \tag{4}$$

The first two terms of the equation 4 are easy to understand. The first represents the original location of the body on the line. The second term tell us that the distance increases (or decreases) every second by v_0 meters due to the original speed v_0. The third term of the equation describes how the acceleration influences the distance. To understand this one, let us consider a body travelling from rest with the constant acceleration a. The original speed of the motion is zero. The final speed of the motion is at. Since the rate of change of the speed is constant, we can use the average speed

$$v_{ave} = \frac{0 + at}{2}$$

to compute the covered distance,

$$d = v_{ave} t = \frac{at}{2} t = \frac{at^2}{2}.$$

Problem 2 *A body is released from rest not too far from the Earth's surface. Fill out the following table where t is time in seconds and $s(t)$ is the distance the body covers in the time period t. Use the space below for your computations, if needed.*

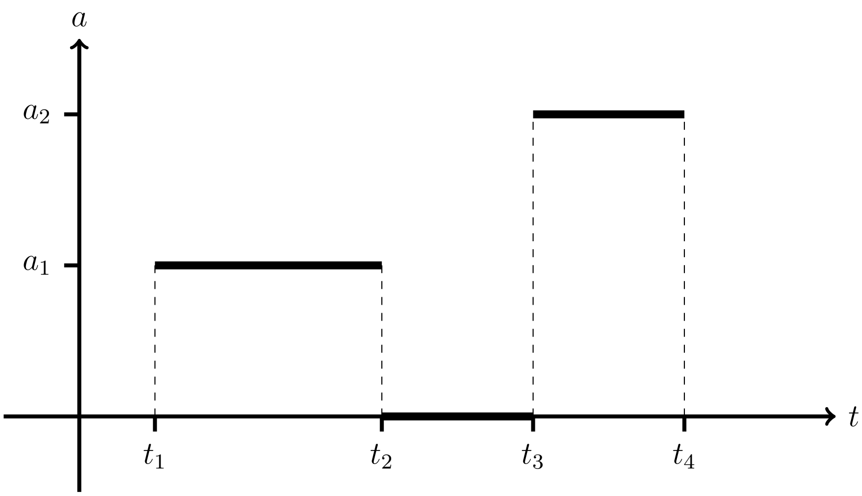
t	*0*	*1*	*2*	*3*
s				

Problem 3 *The free fall acceleration on Europa, a Jovian moon hiding an ocean of liquid water under a thick ice crust, is 1.314* m/s^2*. Standing on the top of a 400-meters-tall Waldorf Astoria Europa hotel, Captain James T. Kirk accidentally dropped his UCLA school ring off the roof. The ring had no initial vertical speed.*

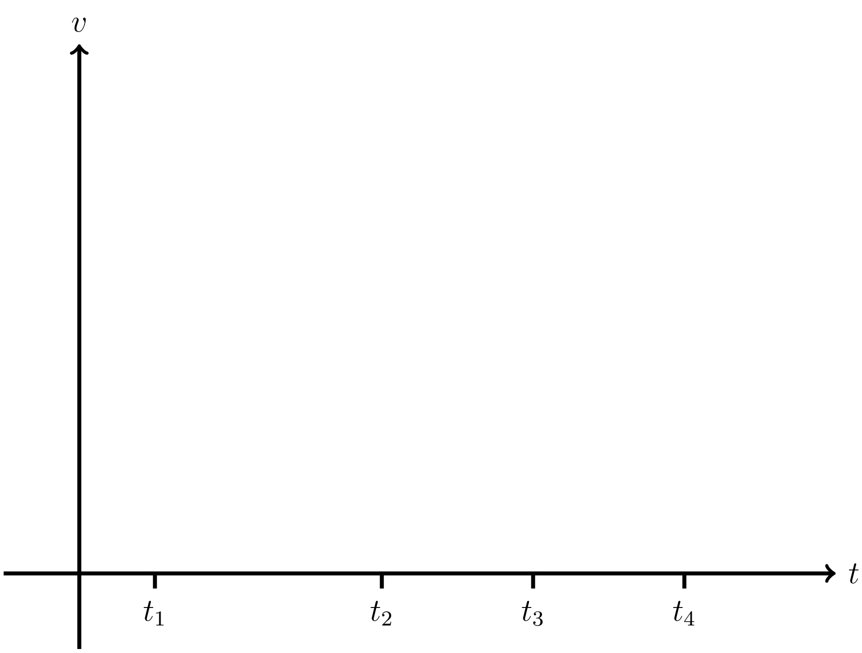
How much time did it take the ring to hit the ground?

How fast was the ring travelling at the moment of the impact?

Problem 4 *The following graph shows acceleration of an object moving from rest along a straight line as a function of time.*



Draw the graph of the speed of the motion as a function of time.



Problem 5 *The free fall acceleration on the Moon is* $1.62\ m/s^2$*. You throw a rock vertically form the surface with the initial speed* $30\ m/s$*. How high will it go?*

You repeat the experiment on Earth. How high would the rock go this time?

We will need to refresh the theory of quadratic equations for what follows.

An equation of the form

$$ax^2 + bx + c = 0 \tag{5}$$

is called *quadratic* if $a \neq 0$. The following quantity is called the *discriminant* of the equation.

$$D = b^2 - 4ac$$

Let us consider 5 over the real numbers. This means that the coefficients a, b, and c of the equation are real and we are only interested in the real-valued solutions x. In this case the following is true.

1. $D < 0$ – no real roots.

2. $D = 0$ – one real root, $x = -b/2a$.

3. $D > 0$ – two real roots, $x_{1,2} = \dfrac{-b \pm \sqrt{D}}{2a}$.

Problem 6 *Solve the equation* $2x^2 - 18x + 36 = 0$.

Problem 7 *You drop from rest a rock into a well and use a stopwatch to measure the time it takes the sound of the impact to come back to you. The time t the stopwatch shows is 10 sec. The speed of sound in the air at the* $20°C$ *temperature is 344 m/s. How deep is the well?*

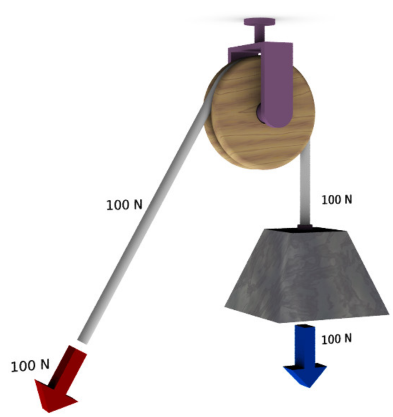
Work of/against a force is defined as a product of the force and of the displacement along/against the direction of the force. For example, lifting an object of mass m to the height h, we work against the Earth's gravitational field.

$$W = mgh \tag{6}$$

Energy is the ability of an object to perform work. For example, a stationary object at height h above the ground has the *potential energy*

$$U = mgh. \tag{7}$$

If we attach the object to an end of a pulley with the advantage one (and disregard friction),



the object would be able to lift a body of the same mass, m, to the same height, h. This type of energy, the energy of the position, is called the *potential energy*.

A different type of energy, called the *kinetic energy* or the energy of motion, is defined as the work needed to accelerate a body of the given mass m from rest to the given speed v. The kinetic energy is described by the following equation.

$$E = \frac{mv^2}{2} \tag{8}$$

To understand formula 8, let us consider a body of mass m accelerating at a constant rate a along a straight line from rest to the speed v. The formula $v = at$ gives us $a = v/t$. According to the Second Law of Newton, the accelerating force $F = ma$. Since the speed of the motion is changing at a constant rate, we can use the formula

$$d = \frac{at^2}{2}$$

to compute the displacement. The work is the product of F and d.

$$W = ma\frac{at^2}{2} = m\frac{a^2t^2}{2} = m\frac{v^2\cancel{t^2}}{2\cancel{t^2}} = \frac{mv^2}{2}.$$

The *Law of the Conservation of Energy* is possibly the most fundamental law of physics. It states that the amount of energy in an isolated system is always preserved. In other words, energy cannot be lost or created. It can transform from one form to another, from potential to kinetic, from chemical to heat, etc., but it cannot disappear or appear from nothing.

The following problem is our first check of the energy conservation law.

Problem 8 *An object of mass m is dropped vertically with no initial speed from the height h.*

- *What is the potential energy U of the object at the initial moment of time?*

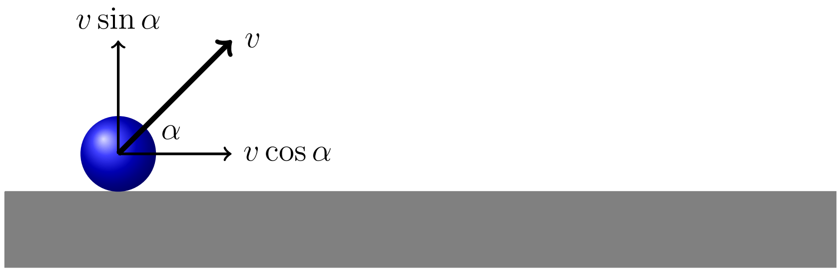
- *What is the speed of the object when it hits the ground? Hint: compare to Problem 3.*

- *What is the potential energy of the object when it hits the ground?*

- *What is the kinetic energy E of the object when it hits the ground?*

- *Is U equal to E?*

An object of mass m is thrown with the original speed v at an angle α from the ground.



We can consider the 2-dimensional motion of the object as a combination of two 1-dimensional motions. The first is the vertical motion with the original speed $v \sin \alpha$. The second is the horizontal motion with the original speed $v \cos \alpha$. Or in vector notations,

$$\vec{v} = (v \cos \alpha, v \sin \alpha), \quad ||\vec{v}|| = v.$$

Problem 9

- *How long will it take the object to reach the maximal height?*

- *How long will it take the object to hit the ground?*

- *What horizontal distance $d(\alpha)$ will the object cover if $v = 100\ m/s$? Use the trigonometric identity*

$$2 \sin \alpha \cos \alpha = \sin 2\alpha$$

to simplify the expression for $d(\alpha)$.

- *Find the values of $d(\alpha)$ for the angles α given below and fill out the following table.*

α	10°	20°	30°	40°	50°	60°	70°	80°	90°
d									

Problem 10 *What is the maximal value for the function*

$$y = \sin x$$

over the real numbers?

Problem 11 *Disregarding air resistance, a.k.a. drag, at what angle α should one launch a projectile from the ground level so that it covers the maximal possible distance?*

Problem 12 *The muzzle speed of the M107 high explosive (HE) shell fired from the M777 howitzer using Charge 8 super is 827 m/s. Disregarding drag, what is the maximal range of the weapon? Compare the result of your computation to the gun's effective maximum range of 24 km. Why do you think is the difference so huge?*

M777 Light Towed Howitzer in service with the 10th Mountain Division in Support of Operation Enduring Freedom, Logar Province, Charkh District, Afghanistan. The photograph taken by Jonathan Mallard.

Problem 13 *Two kids are playing throwing a ball to each other. What is the maximal height the ball reaches, if it takes the ball 2 s. to cover the distance between the children?*

Problem 14 *A ball is thrown vertically down from the height* $h = 1\ m$ *at some speed* v. *It hits the ground and bounces to the height* $2h$. *Find* v. *Ignore drag and the energy lost in the process of bouncing.*

Problem 15 *A body falling from rest has passed the last 30 m of its path in 0.5 s. What was the height of the fall?*

Problem 16 *It took* $t_1 = 1\ s$ *for the first car of a train to pass an observer standing on a platform. It took* $t_2 = 1.5\ s$ *for the second car to pass the observer. Each car was of length* $l = 12m$*. The train was moving at a constant acceleration* a*. Find* a*.*