GEOMETRY USING POINT MASSES

MATH CIRCLE (HS1) 4/19/2015

Today we will be studying **point masses** (that is, adding a mass/weight to a point). Given a point P, we will use $m(P)$ to denote the mass at point P (if there is one).

If we have two points P, Q, view the line segment PQ as a lever (or seesaw). Archimedes famous "Law of the Lever" tells us the lever will balance at a point R if

$$|PR| \cdot m(P) = |QR| \cdot m(Q).$$

Such a point of balance is called a **center of mass**. It also makes sense to talk about center of mass for more than 2 points.

We have the following properties

(1) Any finite set of point masses has a center of mass, which is unique.
(2) For two point masses, their center of mass lies on the segment joining the points and dividing the segment in the ratio which is inversely proportional to the corresponding masses. (This is a restatement of the "Law of the Lever" given above.)
(3) The position of the center of mass of a system of point masses is not changed by replacing several point masses from the system with their total mass positioned at the center of mass of this subsystem. (Simpliest useful case: suppose P, Q, R are point masses, and that the center of mass of P and Q is S. Then the center of mass of the system P, Q, R is the same as the center of mass of R and S with $m(S) = m(P) + m(Q)$.)

Warmups

1) Suppose P, Q, R is a system of point masses with the Q on the line segment PR.

a) Prove that the center of mass Z of this system is also on the line segment PR.

b) Find the ratio$|PZ| : |ZR|$, if $m(P) = 2, m(Q) = 9, m(R) = 4$ and $|PQ| : |QR| = 1 : 3$.

2) Suppose $ABCD$ is a rectangle. Suppose $m(A) = 3, m(B) = m(D) = 2, m(C) = 5$. Let Z be the center of mass.

a) Prove that the center of mass of this system lies on the diagonal AC.

b) Find the ratio $|AZ| : |ZC|$.

Applications to Geometry

1) a) Prove that all the medians in a triangle meet at a single point.

b) Prove that the point above divides each median into pieces with a ratio of $2 : 1$.

This handout is based on the previous handout on point masses by Olga Radko.

2) Let $ABCD$ be a convex quadrilateral. Let K, L, M, N be the midpoints of AB, BC, CD, DA respectively. Suppose O is the intersection of KM and LN.

a) Show O is the midpoint of KM and LN.

b) Show O is also the midpoint of PQ, where P, Q are the midpoints of the diagonals AC, BD.

c) Show $KLMN$ is a parallelogram.

3) Let AD be the median bisecting the side BC in $\triangle ABC$. Let Z be the midpoint of AD. Find the ratio in which the line going through B, Z divides the side AC.

4) Let M be the point on the side AC of $\triangle ABC$ such that $|AM| : |AC| = 1 : 3$. Let N be on the line BC such that B is the midpoint of NC. Let P be the point of intersection of AB with MN. Find the ratios $|AP| : |PB|$ and $|MP| : |PN|$.

5) Let $ABCD$ be a quadrilateral such that a circle can be insribed into it. Let M, N, P, Q be the points where the sides (AB, BC, CD, DA respectively) of the quadrilateral are tangent to the circle. Suppose that $|AM| = a, |BN| = b, |CP| = c, |DQ| = d$.

a) Find (with geometric proof) $|MB|, |NC|, |PD|, |QA|$.

b) Let Z be the intersection of MP and NQ. Find the ratios $|MZ| : |ZP|$ and $|QZ| : |ZN|$.

c) Prove the "Intersecting Chords Theorem": Let ST and UV be two chords intersecting at W. Then $|SW| \cdot |WT| = |UW| \cdot |WV|$.

6) Let $ABCD$ be a parallelogram. Let K be a point on line segment AB such that $|AK| : |AN| = 1 : n$. In what ratio does the line DK divide the diagonal AC.

7) A line goes through the vertex A of a triangle ABC and the midpoint L of the median BB'. In what proportion does this line divide the median CC'.

Extras

1) Develop an analytic geometry approach to center of mass. (As a start, if point $P = (a, b)$ with mass m and point $Q = (c, d)$ with mass n, can you find an equation for the center of mass? What about for systems with more than 2 points?) Use your approach to redo Problem 1 in Applications to Geometry.

2) Suppose we replaced Property (2) at the start of the handout with the following (call it (2')):

- The center of mass of two point masses lies on the segment joining the two points.
- The center of mass of two point masses with equal mass is their midpoint. If mass is added to one of the point masses, the center of mass will be closer to that point (than the other). Similarly, if mass is taken from one of the point masses, the center of mass will be farther from that point (than the other).

Prove Property (2) from (1),(2'),(3). Hint: This is probably asking too much. First try to prove the result for integer point masses. Answering the following intermediate question could be useful: "Suppose we have N point masses of equal mass equally spaced along a line. What is their center of mass?"