

Line Geometry

April 19th, 2015

Warm Up Problem¹

Albert and Bernard just met Cheryl. "When's your birthday?" Albert asked Cheryl.

Cheryl thought a second and said, "I'm not going to tell you, but I'll give you some clues." She wrote down a list of 10 dates:

May 15, May 16, May 19

June 17, June 18

July 14, July 16

August 14, August 15, August 17

"My birthday is one of these," she said.

Then Cheryl whispered in Albert's ear the month - and only the month - of her birthday. To Bernard, she whispered the day, and only the day.

"Can you figure it out now?" she asked Albert.

Albert: I don't know when your birthday is, but I know Bernard doesn't know, either.

Bernard: I didn't know originally, but now I do.

Albert: Well, now I know, too!

When is Cheryl's birthday?

July 16

¹From The New York Times

The following notation is used when solving problems with lines and line segments:

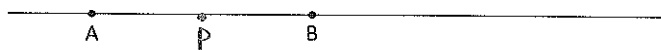
- A, B, C, D, \dots denote points;
- AB denotes a segment with end points A and B ;
- $|AB|$ denotes the length of a segment AB ;
- l denotes a line.

Distances on a straight line

Draw a picture for every problem, if already not drawn.

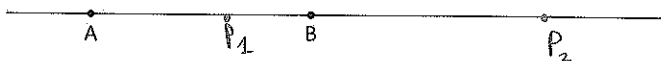
1. Let l be a straight line going through points A and B .

- (a) Find all points P on the line l such that $|AP| = |BP|$. How many such points are there?



1 point

- (b) Find all points P on the line l such that $|AP| = 2|BP|$. How many such points are there?



2 points

- Find n such that: $|AB| = n \times |BP|$

$$P_1: |AB| = |AP_1| + |P_1B|$$

$$\text{So, } |AB| = 2|BP_1| + |BP_1|$$

$$\text{So, } |AB| = 3|BP_1|$$

$$n = 3$$

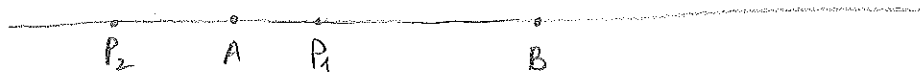
$$P_2: |AP_2| = |AB| + |BP_2|$$

$$\text{So, } 2|BP_2| = |AB| + |BP_2|$$

$$\text{So, } |BP_2| = |AB|$$

$$n = 1$$

- (c) Find all points P on the line l such that $|AP| = \frac{1}{3}|BP|$. Draw a picture similar to the previous parts of the problem. How many such points are there?



- For each solution, find n such that: $|AB| = n \times |BP|$.

$$\begin{aligned} P_1: |AP_1| + |P_1B| &= |AB| \\ \frac{1}{3}|BP_1| + |BP_1| &= |AB| \\ \text{So, } |AB| &= \frac{4}{3}|BP_1| \end{aligned}$$

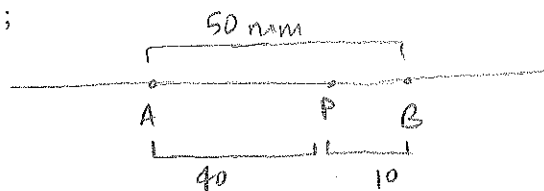
$$n = 4/3$$

$$\begin{aligned} P_2: |AP_2| + |AB| &= |BP_2| \\ \frac{1}{3}|BP_2| + |AB| &= |BP_2| \\ |AB| &= \frac{2}{3}|BP_2| \end{aligned}$$

$$n = 2/3$$

2. Let $|AB| = 50$ mm. Let P be a point inside of AB . Find the values of $|AP|$ and $|BP|$ for the following cases.

- (a) $|AP| = 4|BP|$;

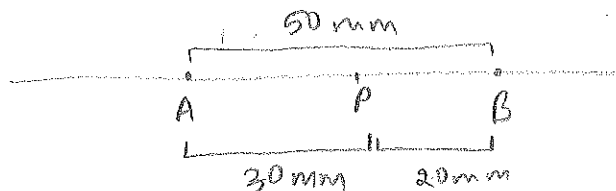


$$|AB| = 50 \text{ mm}$$

$$|AP| = 40 \text{ mm}$$

$$|BP| = 10 \text{ mm}$$

- (b) $2|AP| = 3|BP|$;

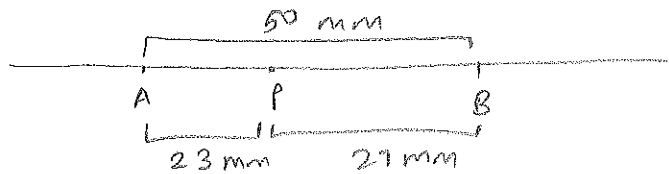


$$|AB| = 50 \text{ mm}$$

$$|BP| = 20 \text{ mm}$$

$$|AP| = 30 \text{ mm}$$

(c) $\frac{|AP|}{|BP|} = \frac{23}{27}$;

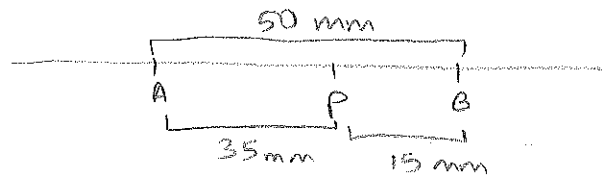


$|AB| = 50 \text{ mm}$

$|BP| = 27 \text{ mm}$

$|AP| = 23 \text{ mm}$

(d) $|AP| - |BP| = 20$;



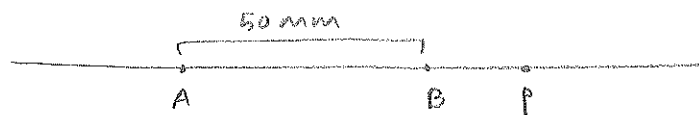
$|AB| = 50 \text{ mm}$

$|BP| = 15 \text{ mm}$

$|AP| = 35 \text{ mm}$

3. Let $|AB| = 50 \text{ mm}$. Let P be a point lying on the line through A and B , but outside of the segment AB .

(a) Let $|AP| = 6|BP|$.



i. Find the value of n in the following expression. $|AB| = n \times |BP|$

$$|AB| + |BP| = |AP|$$

$$|AB| + |BP| = 6|BP|$$

$$|AB| = 5|BP|$$

$$n = 5$$

- ii. Use your answer to find the values of $|AP|$ and $|BP|$.

$$|AB| = 5|BP|$$

$$|AB| = 50 \text{ mm}, \text{ so, } |BP| = 10 \text{ mm}$$

$$|AP| = 6|BP|$$

$$\text{so, } |AP| = 60 \text{ mm.}$$

- (b) Let $7|AP| = 2|BP|$.

- i. Is P closer to B or to A ?

$$|AP| = \frac{2}{7}|BP|. \text{ } AP \text{ is smaller than } BP.$$

So, P is closer to A .

- ii. On which side of AB does point P lie?

On the left side, to the left of A .

- iii. Draw a picture below.



- iv. Look at your picture to justify that $|BP| = |AB| + |AP|$. Fill in the blanks in the following expression.

$$2|BP| = \boxed{2}|AB| + \boxed{2}|AP|$$

- v. Knowing that $7|AP| = 2|BP|$, replace the $2|BP|$ term with the expression you determined above and complete the following expression.

$$\boxed{7}|AP| = \boxed{2}|AB| + \boxed{2}|AP|$$

- vi. Therefore, we can conclude that:

$$\boxed{5}|AP| = \boxed{2}|AB|$$

vii. What are the values of $|AP|$ and $|BP|$?

$$5|AP| = 2|AB|, |AB| = 50 \text{ mm}$$

$$\text{So, } 5|AP| = 100 \text{ mm}$$

$$|AP| = 20 \text{ mm}$$

$$|BP| = \frac{7}{2}|AP|, \text{ so, } |BP| = 70 \text{ mm}$$

(c) Let $\frac{|AP|}{|BP|} = \frac{1}{4}$. Use the steps from part (b) to determine the values of $|AP|$ and $|BP|$.



P lies on the left side of A
because P is closer to A.

$$|BP| = |AP| + |AB|$$

$$\text{because } 4|AP| = |BP|$$

$$4|AP| = |AP| + |AB|$$

$$3|AP| = |AB|$$

$$|AB| = 50 \text{ mm, so, } |AP| = \frac{50}{3} \text{ mm}$$

$$|BP| = 4|AP|$$

$$\text{So, } |BP| = 4 \times \frac{50}{3} \text{ mm} = \frac{200}{3} \text{ mm}$$

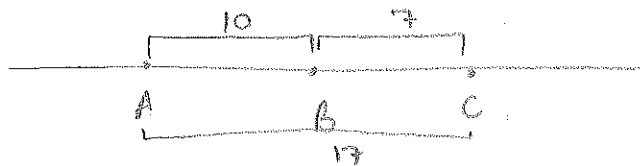
(d) What must $|AP| - |BP|$ equal? Draw a picture to show why this is true.



$|AP| - |BP|$ must equal
the length of AB,
or $|AB|$.



4. Points A, B, C lie on a line l . Find all possible values of $|AC|$ if $|AB| = 10$ and $|BC| = 7$.
(Hint: Draw a picture.)



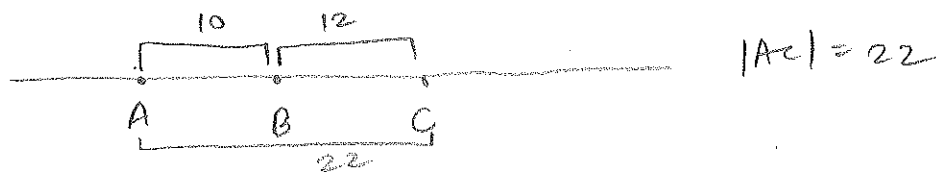
$$|AC| = 17$$

$$|AC| = 3$$

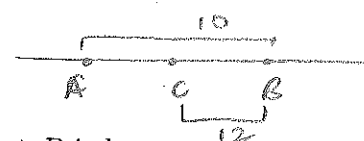
Not possible:



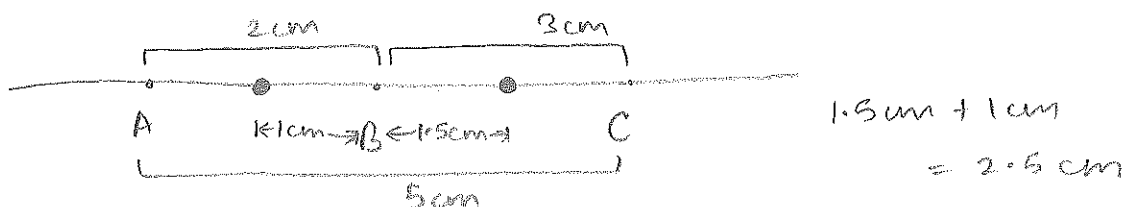
5. Points A , B , C lie on a line l . Find all possible values of $|AC|$ if $|AB| = 10$ and $|BC| = 12$.



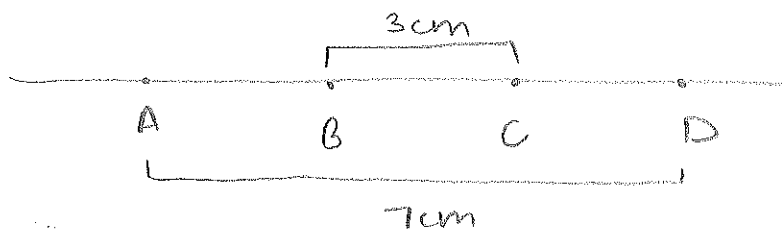
Not possible:



6. Points A , B , and C lie on a line l . $|AB| = 2\text{cm}$, $|AC| = 5\text{cm}$, and point B is between A and C . Find the distance between the midpoints of segments AB and BC .



7. Points A , B , C and D lie on a straight line (in this order). $|AD| = 7\text{cm}$ and $|BC| = 3\text{cm}$. Find the distance between the midpoints of the segments AB and CD .



$$AB + BC + CD = 7\text{cm}$$

$$AB + 3 + CD = 7\text{cm}$$

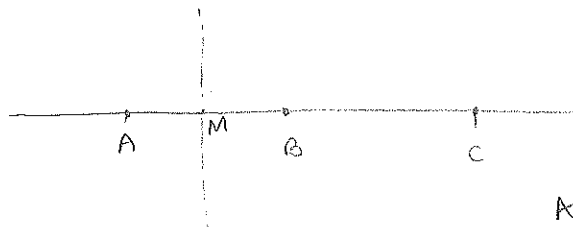
$$AB + CD = 4\text{cm}$$

7 So, $\frac{AB}{2} + \frac{CD}{2} = \frac{4}{2} = \underline{\underline{2\text{cm}}}$

So, $2 + 3 = 5\text{cm}$

8. Let A , B and C be points lying on a line l in this order. Mark all points P on this line, such that

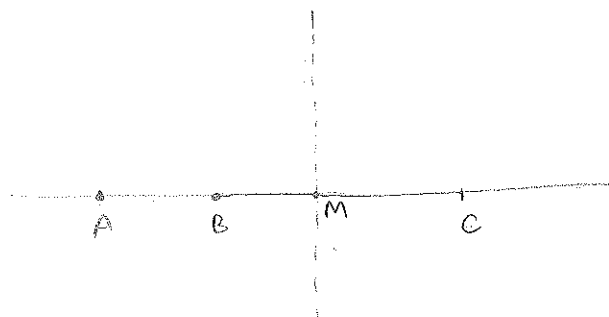
- P is closer to B than to A ;



$$|AM| = |MB|$$

Any point to the right of M .

- P is closer to C than to A .

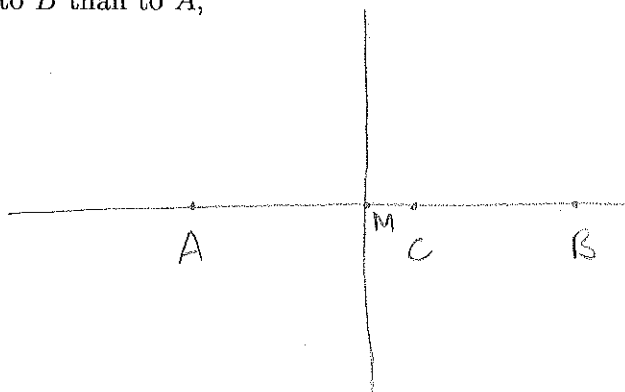


$$|AM| = |MC|$$

Any point to the right of M .

9. Solve the same problem for the case when B is outside of the segment AC .

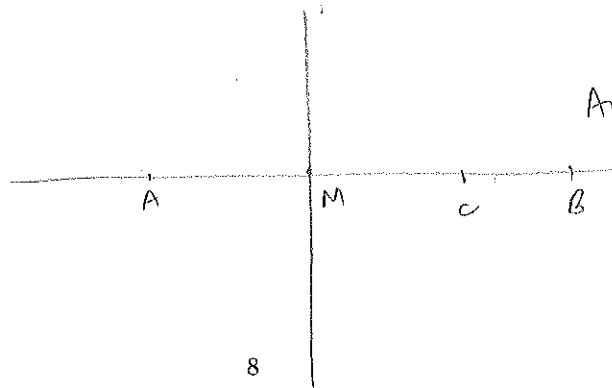
- P is closer to B than to A ;



$$|AM| = |MB|$$

Any point to the right of M .

- P is closer to C than to A .



$$|AM| = |MC|$$

Any point to the right of M .

10. Two segments AB and CD lie on the same line in such a way that their midpoints coincide. Show that $|AC| = |BD|$.



M is the midpoint of AB and CD .

$$\begin{aligned} |AM| &= |MB| \\ |AC| + |CM| &= |MD| + |DB| \\ \text{So, } |AC| &= |DB| \end{aligned}$$

11. There are 25 segments on a line positioned in such a way that they are covering a segment of length 13. Show that at least one segment has length at least equal to 0.5. Assume that the line segments do not coincide.

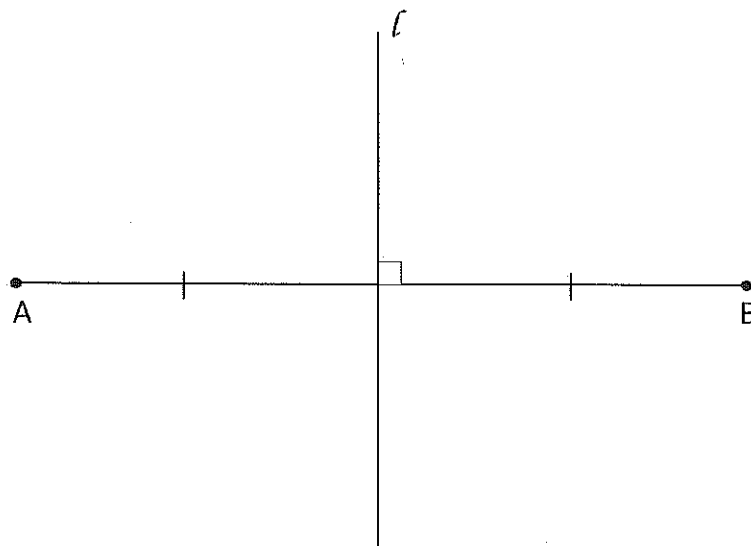
$$\frac{13}{25} > 0.5$$

So, if we wish to split the segment of length 13 into 25 parts, at least one must be equal to 0.5 at least

Two Dimensions

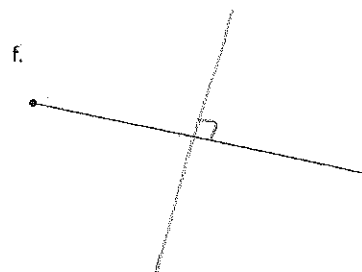
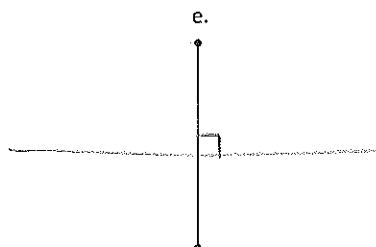
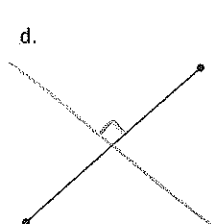
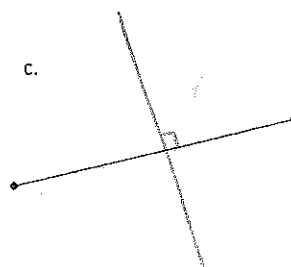
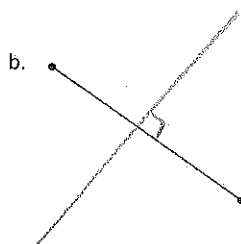
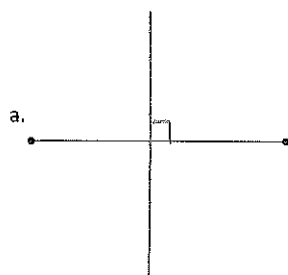
Perpendicular Bisectors

A perpendicular bisector is a line that is perpendicular to and passes through the midpoint of a line segment.

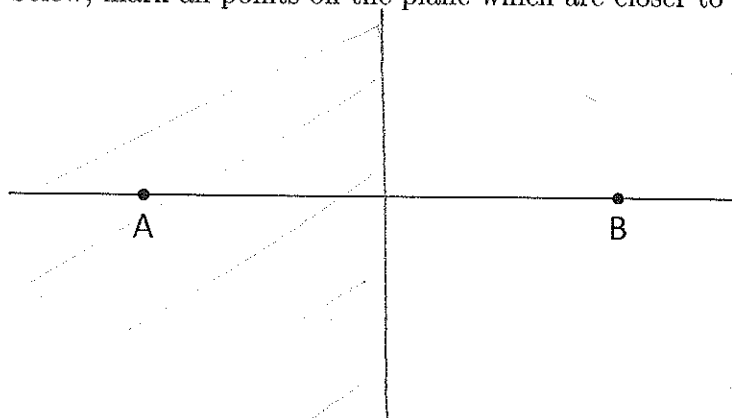


- Points on the perpendicular bisector have the property of being at the same distance from A and B.

1. Draw perpendicular bisectors to each of the segments below.

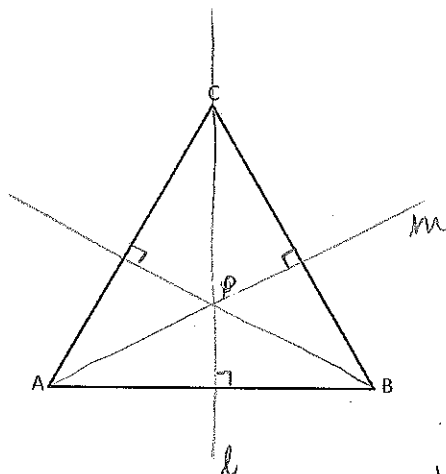


2. On the picture below, mark all points on the plane which are closer to A than to B.



3. Below is triangle ABC . Use perpendicular bisectors to find all points P on the plane such that $|AP| = |BP| = |CP|$. How many such points are there?

(Hint: First, find all points P such that $|AP| = |BP|$. Then find all points P such that $|BP| = |CP|$.)

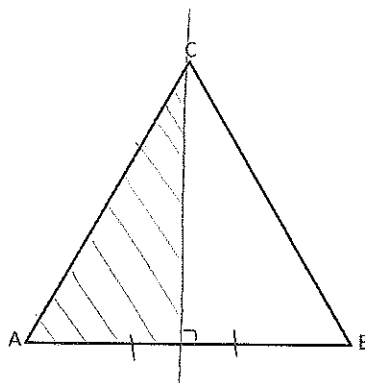


Points P such that $|AP| = |BP|$ lie on line l .

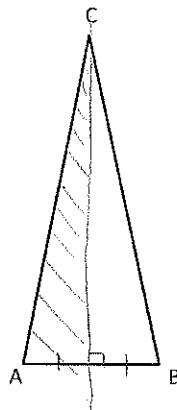
Points P such that $|BP| = |CP|$ lie on line m .
 m and l intersect at P .

So, where all perpendicular bisectors intersect,

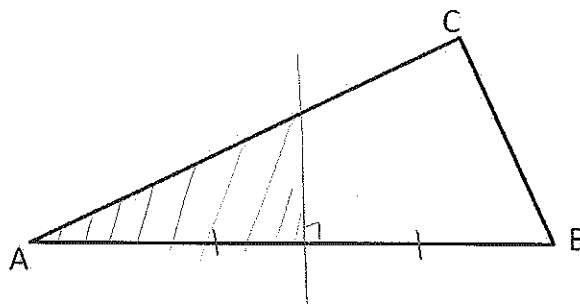
4. Show all the points inside triangle ABC that are closer to vertex A than B. $|AP| = |BP| = |CP|$.



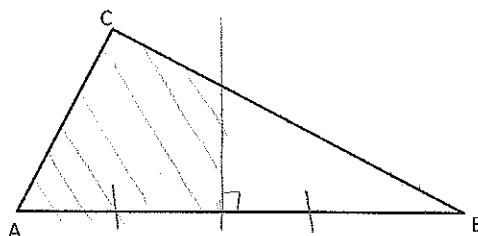
5. Show all the points inside triangle ABC that are closer to vertex A than B .



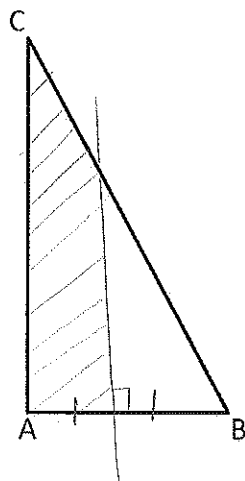
6. Show all the points inside triangle ABC that are closer to vertex A than B .



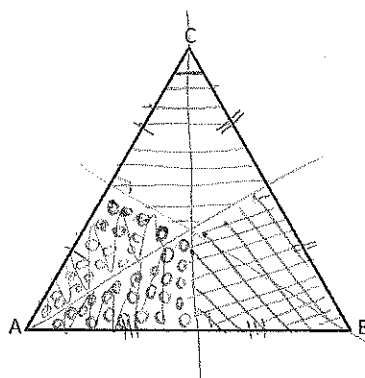
7. Show all the points inside triangle ABC that are closer to vertex A than B .



8. Show all the points inside triangle ABC that are closer to vertex A than B .



9. Show all the points inside triangle ABC that are closer to vertex B than any of the other vertices. Color the region which has all such points. Similarly, use different colors to shade the regions which have points closest to A and C , respectively.



≡ closest to C
 ○○○ closest to A
 ≡≡≡ closest to B

