

GEOMETRY: CIRCUMSCRIBING POLYGONS

MATH CIRCLE (HS1) 4/12/2015

Unless specified otherwise, everything today is to be proven geometrically.

In most cases, all you will need to know are basics about triangles and similar triangles.

The following about arcs on circles may also be useful.

Definition:

Given a circular arc \widehat{AB} we use $\angle\widehat{AB}$ to denote the angle $\angle AOB$, where O is the center of the circle. (Note there is a little ambiguity here, what is it?)

Facts:

- Suppose E is a point on the circle not contained in the arc \widehat{AB} , then $\angle AEB = \frac{1}{2}\angle\widehat{AB}$.
- Suppose E is a point inside the circle. Let AE, BE be extended to chords AC, BD respectively. Then $\angle AEB = \frac{1}{2}(\angle\widehat{AB} + \angle\widehat{CD})$.
- Suppose E is a point outside the circle. Let AE, BE intersect the circle at B, D respectively. Then $\angle AEB = \frac{1}{2}(\angle\widehat{AB} - \angle\widehat{CD})$.

Warmups

0) Given a point, can you find a circle containing that point? Can you freely choose the center of the circle as well? What about the radius? What about both?

1) Given two points, can you find a circle containing both points? Can you freely choose the center of the circle as well? What about the radius? What about both?

2) Given three points, can you always find a circle containing all three points? Can you freely choose the center of the circle as well? What about the radius?

3) Given four points, can you always find a circle containing all four points?

Extra) Explore 0), 1), and 2) algebraically by finding equations for the following (recall a circle with radius r and center (h, k) has equation $(x - h)^2 + (y - k)^2 = r^2$):

a) Circles which contain the point $(0, 0)$.

b) Circles which contain the points $(0, 0)$ and $(2, 0)$.

c) Circles which contain the points $(0, 0)$, $(2, 0)$, and $(3, 1)$.

Triangles

Recall, a circle is uniquely determined by its radius and its center.

We say a circle circumscribes a triangle ABC if the vertices A, B, C all lie on the circle.

We will use the notation $|AB|$ to denote the length of AB .

0) Suppose we have a circle with center K and AB is a chord on the circle. Prove that the perpendicular bisector of AB goes through K . Hint: You may freely use the fact that the perpendicular bisector is unique.

1) Suppose we have a line segment AB .

a) Suppose $R \geq |AB|/2$. Prove that there is a unique circle of radius R that contains A and B .

b) Suppose K is an arbitrary point on the perpendicular bisector of AB . Prove that there is a unique circle with center K that contains A and B .

2) Given a triangle ABC , prove that there is a unique circle that circumscribes it.

3) Suppose A, B, C are all on a line (i.e. a degenerate triangle). Prove that there is no circle that contains A, B, C (i.e. no circle circumscribes a degenerate triangle).

Quadrilaterals

We call a quadrilateral that can be circumscribed a **cyclic quadrilateral**.

0) Give an example of a cyclic quadrilateral and an example of a noncyclic quadrilateral.

1) Prove that $ABCD$ is a cyclic quadrilateral if and only if the perpendicular bisectors of AB, BC, CD , and AD all intersect at the same point.

2) Prove that the following are all cyclic quadrilaterals:

a) Rectangles

b) Isosceles Trapezoids

c) Kites with two right angles.

Characterizations:

The following are all equivalent (for a quadrilateral $ABCD$):

(1) $ABCD$ is a cyclic quadrilateral.

(2) The perpendicular bisectors of AB, BC, CD , and AD all intersect at the same point.

(3) $\angle A + \angle C = \angle B + \angle D = 180^\circ$

(4) $\angle ABD = \angle ACD$ (and similarly $\angle BAC = \angle BDC$, and $\angle ADB = \angle ACB$, etc.)

(5) Let the diagonals AC and BD intersect at E . Then $|AE| \cdot |EC| = |BE| \cdot |ED|$.

Option 1: Proving the Characterization

Note, we have already proven $(1) \Leftrightarrow (2)$.

1) Prove $(1) \Rightarrow (3), (4)$.

2) Prove $(4) \Leftrightarrow (5)$.

3) Prove $(4) \Rightarrow (3)$.

4) Prove $(3) \Rightarrow (1)$.

5) Convince yourself you have actually proven the characterization on the last page.

Extra) Prove the facts about circular arcs given at the start of the handout.

Option 2: Problem Solving with Cyclic Quadrilaterals

1) Let ABC be an equilateral triangle, and let D, E, F be the feet of the altitudes from A, B, C respectively. How many sets of 4 cyclic points can you find? Prove your answer!

2) (Simson Line) Let ABC be a triangle, and let M be a point on its circumcircle (i.e. A, B, C, M form a cyclic quadrilateral). Let D, E, F be the feet of the perpendiculars from M to BC, CA, AB . Prove that D, E, F are collinear.

3) An acute triangle ABC is given in the plane. The circle with diameter AB intersects altitude CC' and its extension at points M, N (i.e. the line $MCC'N$ is perpendicular to AB). Further, the circle with diameter AC intersects altitude BB' and its extension at points P, Q (i.e. the line $BPP'Q$ is perpendicular to AC). Prove that the points M, N, P, Q lie on a common circle.

4) Let $ABCD$ be a cyclic quadrilateral (with circumcenter O). Suppose AC is perpendicular to BD . Prove that the area of the quadrilaterals $A OCD$ and $A OCB$ are equal.

5) Let $ABCD$ be a convex quadrilateral whose diagonals are orthogonal. Let P be the intersection of the diagonals. Let E, F, G, H be the feet of the perpendiculars from P to AB, BC, DC, DA respectively. Prove that E, F, G, H forms a cyclic quadrilateral.