

Modular Arithmetic Part III: Divisibility Rules

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Warm Up Problems

Determine what the reduced form of 4^{120} in mod 7 is by writing down the powers of 4 in mod 7.

$$4^1 \equiv 4 \pmod{7}$$

$$4^2 \equiv 4^1 \times 4 \equiv 16 \equiv 2 \pmod{7}$$

$$4^3 \equiv 4^1 \times 4^2 \equiv 4 \times 2 \equiv 8 \equiv 1 \pmod{7}$$

$$4^4 \equiv 4^1 \times 4^3 \equiv 4 \times 1 \equiv 4 \equiv 4 \pmod{7}$$

$$4^5 \equiv 4^1 \times 4^4 \equiv 4 \times 4 \equiv 16 \equiv 2 \pmod{7}$$

So, the pattern is 4, 2, 1, ...

$$120 = 0 \pmod{3} \quad [120 \text{ is divisible by } 3] \\ \Rightarrow 4^{120} \equiv 1 \pmod{7}$$

Now determine what 4^{120} is in mod 7 arithmetic by representing 120 as a sum of powers of 2 and then reducing 4 to each of those powers mod 7.

$$120 = 64 + 32 + 16 + 8$$

$$4^{120} = 4^{64} \cdot 4^{32} \cdot 4^{16} \cdot 4^8$$

$$\begin{aligned} \Rightarrow 4^{120} &\equiv 2 \times 4 \times 2 \times 4 \\ &= 64 \\ &\equiv 1 \pmod{7} \end{aligned}$$

$$4^1 \equiv 4 \pmod{7}$$

$$4^2 \equiv 4^1 \times 4^1 \equiv 4 \times 4 = 16 \equiv 2 \pmod{7}$$

$$4^4 \equiv 4^2 \times 4^2 \equiv 2 \times 2 \equiv 4 \pmod{7}$$

$$4^8 \equiv 4^4 \times 4^4 \equiv 4 \times 4 \equiv 2 \pmod{7}$$

$$4^{16} \equiv 4^8 \times 4^8 \equiv 2 \times 2 \equiv 4 \pmod{7}$$

$$4^{\text{odd power}} \equiv 4 \pmod{7}$$

$$4^{\text{even power}} \equiv 2 \pmod{7}$$

Divisibility Rules

- Take a number $\underline{a}\underline{b}\underline{c}\underline{d}$ (written with digits a, b, c, d^1). This number is divisible by 2 if and only if

$$\underline{a}\underline{b}\underline{c}\underline{d} \equiv 0 \pmod{2}$$

Instead of dividing the number $\underline{a}\underline{b}\underline{c}\underline{d}$ by 2 and determining if the remainder is 0, we can use modular arithmetic.

- First, write this number as a sum of powers of 10:

$$\underline{a}\underline{b}\underline{c}\underline{d} = a \times 10^3 + b \times 10^2 + c \times 10^1 + d \times 10^0$$

- Next, reduce the powers of 10 in mod 2:

$$10^0 = 1 \equiv 1 \pmod{2}$$

$$10^1 = 10 \equiv 0 \pmod{2}$$

$$10^2 \equiv 0 \pmod{2}$$

$$10^3 \equiv 0 \pmod{2}$$

- Substitute the reduced forms of the powers of 10 into the expansion of $\underline{a}\underline{b}\underline{c}\underline{d}$, and simplify:

$$\begin{aligned} \underline{a}\underline{b}\underline{c}\underline{d} &= a \times 10^3 + b \times 10^2 + c \times 10^1 + d \times 10^0 \equiv a \times 0 + b \times 0 + c \times 0 + d \times 1 \\ &\equiv d \pmod{2} \end{aligned}$$

If d is even, $d \equiv 0 \pmod{2}$

If d is odd, $d \equiv 1 \pmod{2}$

- In mod 2 arithmetic, what is the reduced form of $\underline{a}\underline{b}\underline{c}\underline{d}$?

0 if d is even.

1 if d is odd.

- What is the rule for determining if a number is divisible by 2?

d should be divisible by 2 $\rightarrow d$ should give 0 $\pmod{2}$

¹The digits are underlined to distinguish this expression from the product of the four numbers a, b, c, d

2. Formulate the rule for divisibility of 5 using the same method as divisibility of 2.

(a) Write down \underline{abcd} as a sum of powers of 10:

$$\underline{abcd} = a \times 10^3 + b \times 10^2 + c \times 10^1 + d \times 10^0$$

(b) Reduce the powers of 10 in mod 5 arithmetic.

$$1 \equiv 1 \pmod{5}$$

$$10^1 \equiv 0 \pmod{5}$$

$$10^2 \equiv 0 \pmod{5}$$

$$10^3 \equiv 0 \pmod{5}$$

(c) Substitute the reduced forms of the powers of 10 in mod 5 arithmetic into the expanded form of \underline{abcd} , and simplify the expression:

$$\begin{aligned}\underline{abcd} &= a \times 10^3 + b \times 10^2 + c \times 10^1 + d \times 10^0 \\ &\equiv a \times 0 + b \times 0 + c \times 0 + d \times 1 \pmod{5} \\ &\equiv d \pmod{5}\end{aligned}$$

(d) How can we tell if a number is divisible by 5 without actually dividing by 5? (Note that 0 is divisible by 5).

If d is 0 or 5, $d \equiv 0 \pmod{5}$

So, if a number \underline{abcd} is divisible by 5, d should be 0 or 5.

3. Determine whether the following numbers are divisible by 5 without dividing by 5.

(a) $194825 \equiv 0 \pmod{5}$

Yes.

(b) $37160 \equiv 0 \pmod{5}$

Yes.

(c) $5729 \equiv 4 \pmod{5}$

No.

4. If a number is divisible by 2 and 5, what is the next smallest number it MUST also be divisible by? Why?

10

$$2 \times 5 = 10$$

If a number is divisible by m and n , it is also divisible by mn .

5. A two digit number \underline{ab} can be written as

$$\underline{ab} = 10 \times a + b$$

- (a) Show that \underline{ab} is divisible by 4 if and only if $2 \times a + b$ is divisible by 4. (Use mod 4 arithmetic).

$$10 \equiv 2 \pmod{4} \quad \underline{ab} = 10 \times a + b \\ \underline{ab} \equiv 2 \times a + b \pmod{4}$$

$$\text{If } 2 \times a + b \equiv 0 \pmod{4}$$

$$\text{then } 10 \times a + b \equiv 0 \pmod{4}$$

So, if $2 \times a + b$ is divisible by 4,

then $10 \times a + b$ is divisible by 4.

- (b) Apply this rule to determine if the following numbers are divisible by 4.

• 24

$$2 \times 2 + 4 = 8 \text{ is divisible by 4.}$$

So, 24 is divisible by 4.

• 68

$$2 \times 6 + 8 = 20 \text{ is divisible by 4.}$$

So, 68 is divisible by 4.

• 98

$$2 \times 9 + 8 = 26 \text{ is not divisible by 4.}$$

So, 98 is not divisible by 4.

6. Formulate a rule for divisibility of 4. Please do not start with the number $\underline{a} \underline{b} \underline{c} \underline{d}$. We want to create a rule that applies to all numbers, not just 4 digit numbers. (Hint: You can begin the derivation by reducing powers of 10 in mod 4 arithmetic).

$$10^0 = 1 \equiv 1 \pmod{4}$$

$$10^1 = 10 \equiv 2 \pmod{4}$$

$$10^2 = 10 \times 10^1 \equiv 2 \times 2 \equiv 0 \pmod{4}$$

$$10^3 = 10^1 \times 10^2 \equiv 1 \times 0 \equiv 0 \pmod{4}$$

$$10^4 = 10^1 \times 10^3 \equiv 2 \times 0 \equiv 0 \pmod{4}$$

:

$$10^n \text{ for } n \geq 2 \text{ is } 0 \pmod{4},$$

the divisibility depends on the one's digit and ten's digit.

So, let's say the number ends in $\underline{a} \underline{b}$.

If $\underline{ab} \equiv 0 \pmod{4}$, the number is divisible by 4.

- (a) What are the only digits we need to look at when determining if a number is divisible by 4?

Ten's and one's

(last 2 digits)

- (b) Complete the following sentence: A number is divisible by 4 if and only if

the last two digits are divisible by 4.

7. Determine if the following numbers are divisible by 4.

- 9876543216

Since 16 is divisible by 4,
the number is divisible by 4.

- 6123456789

Since 89 is not divisible by 4,
the number is not divisible by 4.

8. Prove that any even number multiplied by itself is divisible by 4.

Even number multiplied by itself gives an even number.

Even numbers give $2 \pmod{4}$ or $0 \pmod{4}$.

$$0 \times 0 \pmod{4} = 0 \pmod{4}$$

$$2 \times 2 \pmod{4} = 4 \pmod{4} \equiv 0 \pmod{4}$$

Since we always get $0 \pmod{4}$, any even number multiplied by itself is divisible by 4.

9. Prove that any odd number multiplied by itself has a remainder of 1 when divided by 4.

Odd number multiplied by itself gives an odd number.

And in mod 4, odd numbers give $1 \pmod{4}$ or $3 \pmod{4}$

$$1 \times 1 \pmod{4} \equiv 1 \pmod{4}$$

$$3 \times 3 \pmod{4} \equiv 9 \pmod{4} \equiv 1 \pmod{4}$$

So, the remainder is always 1.

10. Formulate the rule for dividing by 25.

$$10^0 \equiv 1 \equiv 1 \pmod{25}$$

$$10^1 \equiv 10 \equiv 10 \pmod{25}$$

$$10^2 \equiv 100 \equiv 0 \pmod{25}$$

$$10^3 \equiv 1000 \equiv 0 \pmod{25}$$

$$10^4 \equiv 10^2 \times 10^2 \equiv 0 \times 0 \equiv 0 \pmod{25}$$

Since 10^n for $n \geq 2$ is $0 \pmod{25}$,

the divisibility depends on the ten's and
one's digits.

So, let's say that the number ends in
ab.

If $ab \equiv 0 \pmod{25}$, the number is
divisible by 25.

So, for divisibility by 25, the last two
digits should be divisible by 25.

11. Determine which of the following numbers are divisible by 25.

• 6150
Yes.

• 82900
Yes.

• 2525252
No.

12. Show that the product of 2 numbers ending in 5 is always divisible by 25. (Hint: Use the fact that each of the numbers is divisible by 5)

If each number is divisible by 5,
the product will be divisible by $5 \times 5 = 25$.

13. Formulate the rule for divisibility by 8.

$$10^0 = 1 \equiv 1 \pmod{8}$$

$$10^1 = 10 \equiv 2 \pmod{8}$$

$$10^2 = 10^1 \times 10^1 \equiv 2 \times 2 \equiv 4 \equiv 4 \pmod{8}$$

$$10^3 = 10^2 \times 10^1 \equiv 4 \times 2 \equiv 8 \equiv 0 \pmod{8}$$

$$10^4 = 10^3 \times 10^1 \equiv 0 \times 1 \equiv 0 \pmod{8}$$

Since 10^n for any $n \geq 3$ gives $0 \pmod{8}$,
the divisibility depends on the last 3 digits.

So, if the last three digits are divisible by 8,
the number is divisible by 8.

14. Determine which of the following numbers are divisible by 8.

• 464

Yes.

• 128

Yes.

• 123456789

No.

15. Knowing that $1000 = 125 \times 8$, determine the rule for divisibility by 125.

$$\text{Since } 1000 \equiv 0 \pmod{125}$$

$$10^4, 10^5, \dots \equiv 0 \pmod{125}$$

So, we only need to check the 3 last digits.

The last three digits should be divisible by 125.

16. Formulate the rule for divisibility by 3.

$$10^0 \equiv 1 \pmod{3}$$

$$10^1 \equiv 10 \equiv 1 \pmod{3}$$

$$10^2 \equiv 1 \times 1 \equiv 1 \pmod{3}$$

$$\therefore \underline{\underline{abc}} = a \times 10^2 + b \times 10^1 + c \times 10^0 \\ \equiv a + b + c \pmod{3}$$

$$\text{Similarly, } \underline{\underline{abcd}} = a \times 10^3 + b \times 10^2 + c \times 10^1 + d \times 10^0 \\ \equiv a + b + c + d \pmod{3}.$$

So, If the sum of the digits gives $0 \pmod{3}$
or the sum of the digits is divisible
by 3, the number is divisible by 3.

17. Determine which of the following numbers are divisible by 3.

- 123456789

$$1+2+3+4+5+6+7+8+9 \\ = 45 \quad \text{Yes.}$$

- 306090

$$3+6+9 = 21$$

Yes.

- 13579

$$1+3+5+7+9 = 25 \\ \text{No.}$$

18. Formulate the rule for divisibility by 9.

$$10^0 \equiv 1 \pmod{9}$$

$$10^1 \equiv 10 \equiv 1 \pmod{9}$$

$$10^2 \equiv 10 \times 10 \equiv 1 \times 1 \pmod{9} \equiv 1 \pmod{9}$$

Similar to the mod 3 arithmetic,

the sum of the digits should give 0 in mod 9.

So, the sum of the digits should be
divisible by 9 for the number to
be divisible by 9.

19. Determine which of the following numbers are divisible by 9.

- 123456789

$$1+2+3+4+5+6+7+8+9 = 45$$

Yes.

- 13579

$$1+3+5+7+9 = 25$$

No.

- 54637281

$$5+4+6+3+7+2+8+1$$

$$11 = 36$$

Yes.

20. Formulate the rule for divisibility by 6.

$$10^0 \equiv 1 \equiv 1 \pmod{6}$$

$$10^1 \equiv 4 \pmod{6}$$

$$10^2 \equiv 4 \times 4 \equiv 4 \pmod{6}$$

$$10^3 \equiv 4 \times 4 \equiv 4 \pmod{6}$$

$$10^4 \equiv 4 \times 4 \equiv 4 \pmod{6}$$

So, If the number is $\underline{\underline{abcd}}$, we should have

$$4(a+b+c) + d \equiv 0 \pmod{6}$$

If the number is $\underline{\underline{\underline{abcde}}}$, we should have

$$4(a+b+c+d) + e \equiv 0 \pmod{6}$$

The sum of the ones digit and 4-times the sum of the rest of the digits should be divisible by 6.

21. How else can you test for divisibility by 6? (Hint: Use the fact that $6 = 2 \times 3$)

If a number is divisible by both 2 and 3,

it must be divisible by 6.

22. Determine which of the following numbers are divisible by 6.

- 546

divisible by 2.

Yes.

$$5+4+6=15 \Rightarrow \text{divisible by 3}$$

- 654

divisible by 2.

Yes.

$$6+5+4=15 \Rightarrow \text{divisible by 3}$$

- 645

not divisible by 2.

No.

23. Show that the product of 3 consecutive numbers is always divisible by 6.

Three consecutive numbers will always have one number divisible by 2 and one number divisible by 3.

So, the product of the three numbers can be divided by 2 and 3.

So, it will always be divisible by 6.

24. Make up your own divisibility test! Use the following guidelines:

- Select a number.
- Reduce the powers of 10 modulo this number (decide when you need to stop!).
- Formulate the rule.

Try the rule for 11.

25. Apply the divisibility rule you came up with in the previous problem to two numbers, one of which is divisible by your number and the other is not.

26. Show that a number written with digits $\underline{x}\underline{y}\underline{z}\underline{w}$ (in this order) is divisible by 99 if and only if the sum of the numbers \underline{xy} and \underline{zw} is divisible by 99. (Hint: Complete the following expression)

$$\underline{x}\underline{y}\underline{z}\underline{w} = \underline{\underline{xy}} \times 100 + \underline{\underline{zw}}$$

$$10^0 \equiv 1 \pmod{99}$$

$$10^1 \equiv 10 \pmod{99}$$

$$100 \equiv 1 \pmod{99}$$

$$\begin{aligned} \text{So, } 100(\underline{\underline{xy}}) + \underline{\underline{zw}} &\equiv 1(\underline{\underline{xy}}) + \underline{\underline{zw}} \\ &\equiv xy + zw \pmod{99} \end{aligned}$$

so, for $\underline{\underline{xy}}\underline{\underline{zw}}$ to be divisible by 99,

$xy + zw$ should give $0 \pmod{99}$,

which means $(xy + zw)$ should be divisible by 99.

27. Is $12^{100} - 10^{100}$ divisible by 11? (Hint: Find the remainders of division of 12^{100} by 11 and 10^{100} by 11).

$$12^1 \equiv 1 \pmod{11}$$

$$12^2 = 1 \times 1 \equiv 1 \pmod{11}$$

$$12^3 = 1 \times 1 \equiv 1 \pmod{11}$$

⋮

so, the remainder of division of 12^{100} by 11 is 1.

$$10^1 = 10 \equiv 10 \pmod{11}$$

$$10^2 \equiv 1 \pmod{11}$$

$$10^3 \equiv 1 \times 10 = 10 \pmod{11}$$

$$10^4 \equiv 1 \times 1 \equiv 1 \pmod{11}$$

⋮

so, the remainder of division of 10^{100} by 11 is 1.

$$\begin{aligned} 12^{100} - 10^{100} &\equiv 1 - 1 \pmod{11} \\ &\equiv 0 \pmod{11} \end{aligned}$$

Yes, it is divisible.

28. If a number multiplied by itself produces a remainder of 1 when divided by 5, what are the possible remainders when the number is divided by 5? (Hint: Since we can do this in mod 5 arithmetic, it is sufficient to look at 0, 1, 2, 3, 4).

$$a \times a \equiv 1 \pmod{5}$$

Guess and check.

$$1 \times 1 \equiv 1 \pmod{5}$$

$$2 \times 2 \equiv 4 \pmod{5}$$

$$3 \times 3 \equiv 9 \equiv 4 \pmod{5}$$

$$4 \times 4 \equiv 16 \equiv 1 \pmod{5}$$

so, to get 1 as the remainder,

a should be 1 or 4.

29. In the following number, cross out the least possible number of digits so that the resulting number is divisible by 36. (Hint: $36 = 4 \times 9$)

5328

65432789

For the number to be divisible by 36, it must be divisible by 4 and 9.

For divisibility by 4, last two digits must be divisible by 4.

For divisibility by 9, sum of digits must be divisible by 9.

30. Find the smallest possible number written with only digits 1 and 0 such that it is divisible by 225. (Hint: Factor 225 to get factors that we know the divisibility rules for)

$225 = 9 \times 25 \rightarrow$ last two digits should be 00
↓
or divisible by 25.
sum must be
divisible by 9

So, 111111100

31. Take the number

$$100! = 1 \times 2 \times 3 \times 4 \times \dots \times 99 \times 100.$$

Add up all of the digits of this number. For the number you get, do the same thing (add up all the digits). Continue to do this until you get a one digit number. What number will you get? (Hint: Think of a divisibility rule which is related to adding up all of the digits of a number).

Since $100!$ is constituted to factors of 9,

it is divisible by 9.

So, the sum of the digits would be
divisible by 9.

The one-digit number divisible by
9 is 9.

So, the number we will get is 9.

We can do the same to say that $100!$ is divisible
by 3, but since $100!$ is also divisible by 9 (by the proof
above) and 9 is greater than 3, we will get 9.

