Modular Arithmetic Part III: Divisibility Rules

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Warm Up Problems

Determine what the reduced form of 4^{120} in mod 7 is by writing down the powers of 4 in mod 7.

$4^1 \equiv 4 \bmod (7)$

$4^2 \equiv 4^1 \times 4 \equiv 16 \equiv 2 \bmod (7)$

$4^3 \equiv$

Now determine what 4^{120} is in mod 7 arithmetic by representing 120 as a sum of powers of 2 and then reducing 4 to each of those powers mod 7.

Divisibility Rules

1. Take a number $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}$ (written with digits a, b, c, d[1]). This number is divisible by 2 if and only if

$$\underline{a}\,\underline{b}\,\underline{c}\,\underline{d} \equiv 0 \pmod 2$$

Instead of dividing the number $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}$ by 2 and determining if the remainder is 0, we can use modular arithmetic.

(a) First, write this number as a sum of powers of 10:

$$\underline{a}\,\underline{b}\,\underline{c}\,\underline{d} = a \times 10^3 + b \times 10^2 + c \times 10^1 + d \times 10^0$$

(a) Next, reduce the powers of 10 in mod 2:

$10^0 = 1 \equiv \qquad \pmod 2$

$10^1 = 10 \equiv \qquad \pmod 2$

$10^2 \equiv \qquad \pmod 2$

$10^3 \equiv \qquad \pmod 2$

(b) Substitute the reduced forms of the powers of 10 into the expansion of $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}$, and simplify:

$$\underline{a}\,\underline{b}\,\underline{c}\,\underline{d} \quad = \quad a \times 10^3 + b \times 10^2 + c \times 10^1 + d \times 10^0 =$$

(c) In mod 2 arithmetic, what is the reduced form of $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}$? .

(d) What is the rule for determining if a number is divisible by 2?

[1]The digits are underlined to distinguish this expression from the product of the four numbers a, b, c, d

2. Formulate the rule for divisibility of 5 using the same method as divisibility of 2.

   (a) Write down $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}$ as a sum of powers of 10:

   (b) Reduce the powers of 10 in mod 5 arithmetic.

   $1 \equiv \qquad \pmod 5$

   $10^1 \equiv \qquad \pmod 5$

   $10^2 \equiv \qquad \pmod 5$

   $10^3 \equiv \qquad \pmod 5$

   (c) Substitute the reduced forms of the powers of 10 in mod 5 arithmetic into the expanded form of $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}$, and simplify the expression:

   (d) How can we tell if a number is divisible by 5 without actually dividing by 5? (Note that 0 is divisible by 5).

3. Determine whether the following numbers are divisible by 5 without dividing by 5.

   (a) 194825

   (b) 37160

   (c) 5729

4. If a number is divisible by 2 and 5, what is the next smallest number it MUST also be divisible by? Why?

5. A two digit number $\underline{a}\,\underline{b}$ can be written as

$$\underline{a}\,\underline{b} = 10 \times a + b$$

(a) Show that $\underline{a}\,\underline{b}$ is divisible by 4 if and only if $2 \times a + b$ is divisible by 4. (Use mod 4 arithmetic).

(b) Apply this rule to determine if the following numbers are divisible by 4.

- 24
- 68
- 98

6. Formulate a rule for divisibility of 4. Please do not start with the number $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}$. We want to create a rule that applies to all numbers, not just 4 digit numbers. (Hint: You can begin the derivation by reducing powers of 10 in mod 4 arithmetic).

   (a) What are the only digits we need to look at when determining if a number is divisible by 4?

   (b) Complete the following sentence: A number is divisible by 4 if and only if

7. Determine if the following numbers are divisible by 4.
   - 9876543216
   - 6123456789

8. Prove that any even number multiplied by itself is divisible by 4.

9. Prove that any odd number multiplied by itself has a remainder of 1 when divided by 4.

10. Formulate the rule for dividing by 25.

11. Determine which of the following numbers are divisible by 25.

- 6150
- 82900
- 2525252

12. Show that the product of 2 numbers ending in 5 is always divisible by 25. (Hint: Use the fact that each of the numbers is divisible by 5)

13. Formulate the rule for divisibility by 8.

14. Determine which of the following numbers are divisible by 8.

- 464
- 128
- 123456789

15. Knowing that $1000 = 125 \times 8$, determine the rule for divisibility by 125.

16. Formulate the rule for divisibility by 3.

17. Determine which of the following numbers are divisible by 3.

- 123456789
- 306090
- 13579

18. Formulate the rule for divisibility by 9.

19. Determine which of the following numbers are divisible by 9.

- 123456789

- 13579

- 54637281

20. Formulate the rule for divisibility by 6.

21. How else can you test for divisibility by 6? (*Hint:* Use the fact that $6 = 2 \times 3$)

22. Determine which of the following numbers are divisible by 6.

- 546

- 654

- 645

23. Show that the product of 3 consecutive numbers is always divisible by 6.

24. Make up your own divisibility test! Use the following guidelines:

- Select a number.
- Reduce the powers of 10 modulo this number (decide when you need to stop!).
- Formulate the rule.

25. Apply the divisibility rule you came up with in the previous problem to two numbers, one of which is divisible by your number and the other is not.

26. Show that a number written with digits $\underline{x}\,\underline{y}\,\underline{z}\,\underline{w}$ (in this order) is divisible by 99 if and only if the sum of the numbers $\underline{x}\,\underline{y}$ and $\underline{z}\,\underline{w}$ is divisible by 99. (*Hint:* Complete the following expression)

$\underline{x}\,\underline{y}\,\underline{z}\,\underline{w} = \_\_\_\_ \times 100 + \_\_\_\_$

27. Is $12^{100} - 10^{100}$ divisible by 11? (*Hint:* Find the remainders of division of 12^{100} by 11 and 10^{100} by 11).

28. If a number multiplied by itself produces a remainder of 1 when divided by 5, what are the possible remainders when the number is divided by 5? (*Hint:* Since we can do this in mod 5 arithmetic, it is sufficient to look at 0, 1, 2, 3, 4).

29. In the following number, cross out the least possible number of digits so that the resulting number is divisible by 36. (*Hint:* $36 = 4 \times 9$)

$$65432789$$

30. Find the smallest possible number written with only digits 1 and 0 such that it is divisible by 225. (*Hint:* Factor 225 to get factors that we know the divisibility rules for)

31. Take the number

$$100! = 1 \times 2 \times 3 \times 4 \times \ldots \times 99 \times 100.$$

Add up all of the digits of this number. For the number you get, do the same thing (add up all the digits). Continue to do this until you get a one digit number. What number will you get? (Hint: Think of a divisibility rule which is related to adding up all of the digits of a number).