LAMC Intermediate I & II **March 1, 2015**

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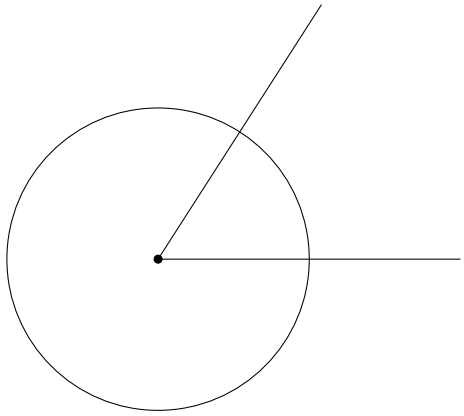
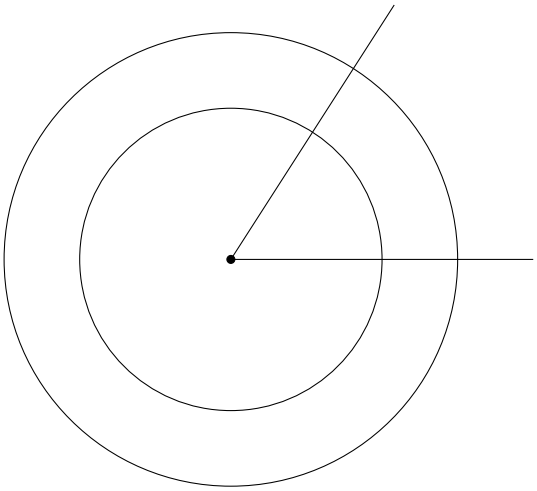
A natural unit of measuring angles, a radian

Consider an angle and a circle such that the vertex of the angle is the center of the circle. If the length of the arc the angle cuts off the circle is equal in length to the circle's radius, we say that the size of the angle is one *radian*.



The definition does not depend on the radius of the circle. Any two concentric circles are similar. Therefore, the ratio of the arc length to the radius is always the same.



It is known that in the Euclidean plane, the length of a circumference of radius r is $2\pi r$, where $\pi = 3.1415926535897932...$ Therefore, there are 2π radians in the 360° angle.

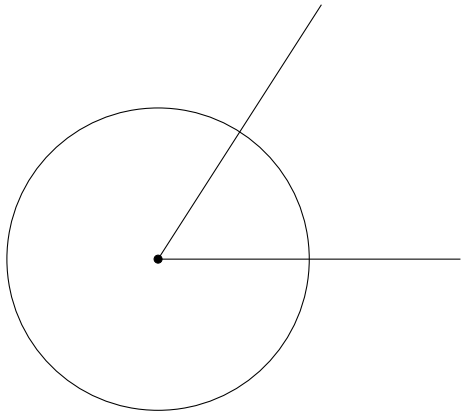
Problem 1 *Convert the following angles to radians.*

15°	30°	45°	60°	90°	120°	180°	270°

Problem 2 *Find the angles, in radians, of a right triangle with an angle* $\pi/6$.

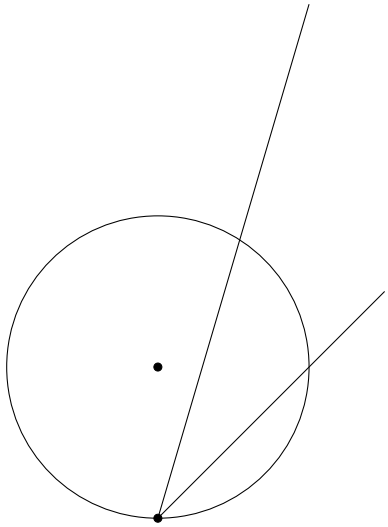
Problem 3 *Two angles of a triangle in the Euclidean plane equal to* $\pi/4$ *and* $2\pi/3$. *Find the size of the third angle in radians.*

An angle having its vertex in the center of a circle is called a *central* angle.



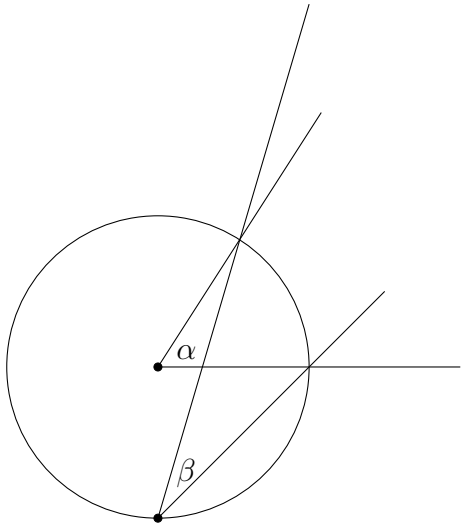
The size of a central angle, in radians, equals to the length of the arc it *subtends*, or cuts off the circle, measured in the circle's radii.

An angle having its vertex on the circumference of a circle is called an *inscribed* angle.



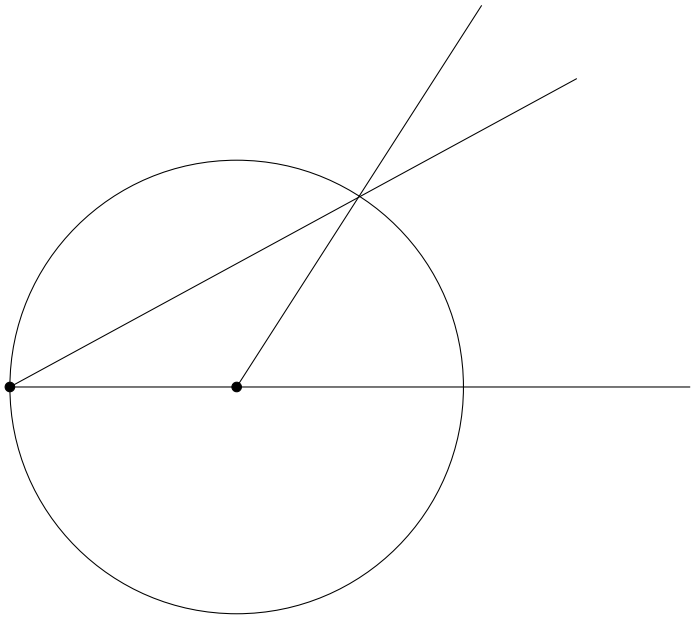
Theorem 1 *If an inscribed and a central angle subtend the same arc on the circle, then the size of the central angle is twice the size of the inscribed one.*

$\alpha = 2\beta$

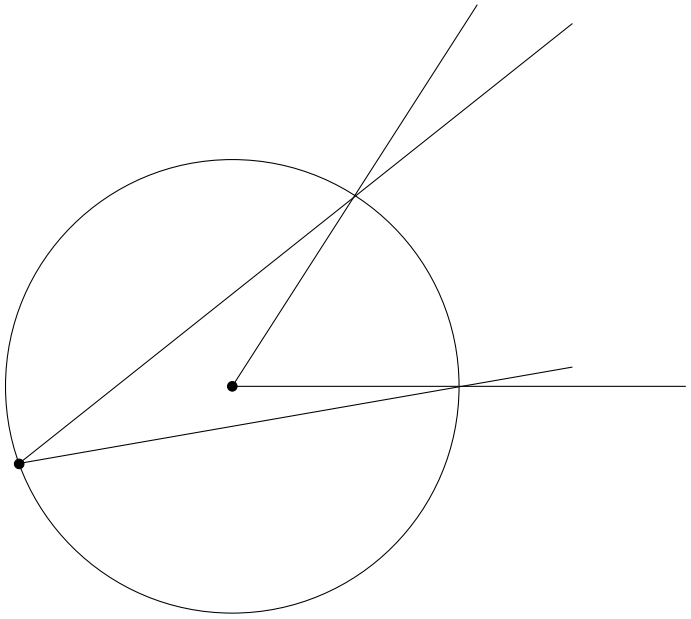


Problem 4 *Recall Thales' theorem. Draw the corresponding picture in the space below. Explain why the theorem of Thales is a particular case of Theorem 1.*

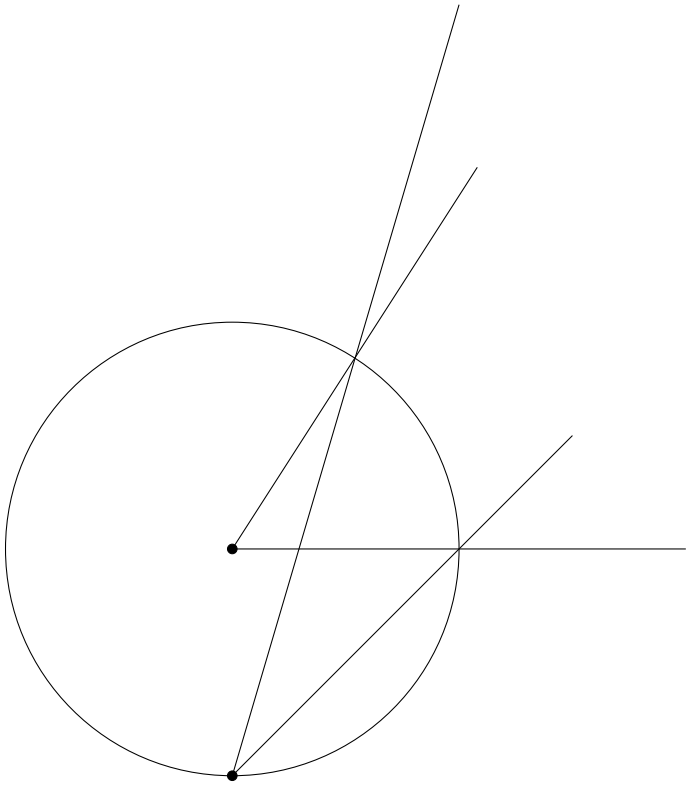
Problem 5 *Prove Theorem 1 in the special case when a side of the inscribed angle passes through the center of the circle. If you need to perform computations involving the angles' sizes, please use radians, not degrees.*



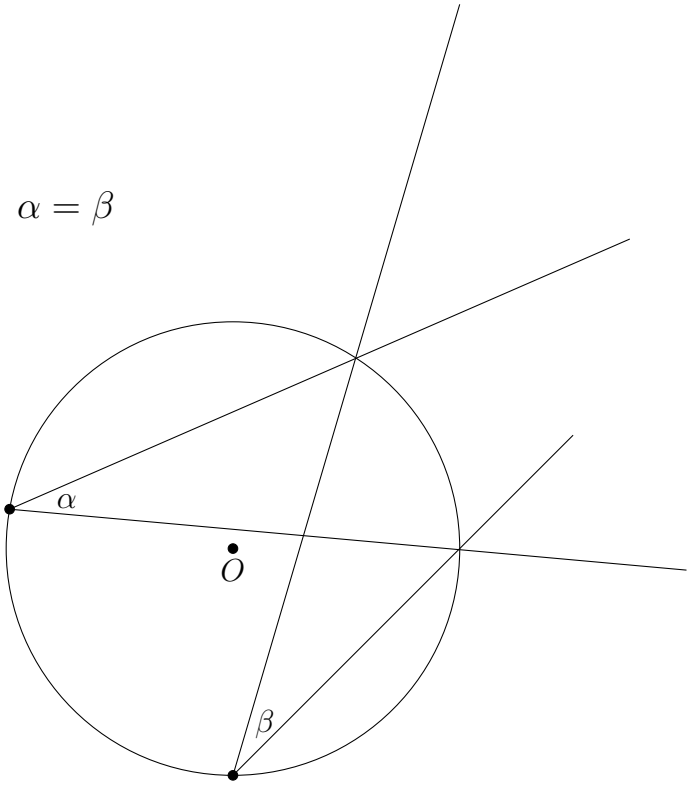
Problem 6 *Prove Theorem 1 in the case when the vertex of the central angle lies inside the inscribed angle.*



Problem 7 *Prove Theorem 1 in the case when the vertex of the central angle does not lie inside the inscribed angle.*

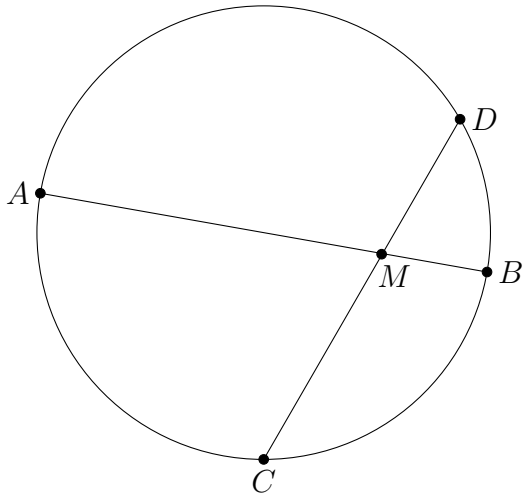


Theorem 2 *Any two inscribed angles subtending the same arc on a circle are of equal size.*



Problem 8 *Prove Theorem 2.*

Theorem 3 *Let AB and CD be any two chords of a circle intersecting at the point M. Then $|AM| \times |MB| = |CM| \times |MD|$.*



Problem 9 *Prove Theorem 3*

A straight line segment *bisector* is a straight line passing through the middle of the segment. A *perpendicular bisector* is the bisector forming the right angle with the original line.

Problem 10 *Prove that a perpendicular bisector is the set of all the points in the plane equidistant from the ends of the segment. (Note that you need to prove two statements, not one!)*

Theorem 4 *Perpendicular bisectors of the sides of any triangle in the Euclidean plane have a common point. The point is the center of the circumference circumscribing the triangle.*

The point is called the *circumcenter* of the triangle. The corresponding circle is called the *circumcircle*.

Problem 11 *Prove Theorem 4.*

Theorem 5 *The altitudes of any triangle in the Euclidean plane intersect at one point.*

The point is called the *orthocenter* of the triangle.

We will need the following proposition as a tool to prove Theorem 5.

Proposition 1 *If one takes a triangle in the Euclidean plane and draws a straight line parallel to the opposing side through every vertex of the triangle, then the sides of the original triangle are the midlines of the triangle formed by the straight lines.*

Problem 12 *Prove Proposition 1.*

Problem 13 *Use Theorem 4 and Proposition 1 to prove Theorem 5.*

If you are finished doing all the above, but there still remains some time ...

Problem 14 *While playing in the park, Ivan and Peter came to a large round clearing surrounded by a ring of aspen trees. The boys decided to count the trees. As they walked around the clearing counting, Ivan's 20th tree turned out to be Peter's 7th, while Ivan's 7th was Peter's 94th. How many trees were growing around the clearing?*

Problem 15 *The length of the Earth's equator is 40,075 km. A rope is tied tight around the Equator. Then they add a 1-meter-long piece to the rope and stretch it evenly above the Earth's surface. Assuming that the Earth is a perfect ball, would a cat be able to squeeze itself between the Earth and the rope?*

Problem 16 *A train moves in one direction for 5.5 hours covering any 100-mile part of the journey in 1 hour. Is the train necessarily moving at a constant speed? Is the train's average speed for the entire journey necessarily equal to 100 mph?*