AMC PREP + CATCHUP

MATH CIRCLE (HS1) 1/25/2015

AMC Problems (Taken from 2007 A&B Tests)

Note: Remember that the real AMC test includes multiple choice questions. When doing these problems, ask yourself if having possible answers could be used to your advantage.

1) Last year Mr. John Q. Public received an inheritance. He paid 20% in federal taxes on the inheritance, and paid 10% of what he had left in state taxes. He paid a total of $\$10,500$ for both taxes. How many dollars was the inheritance?

2) Triangles ABC and ADC are isosceles with $AB = BC$ and $AD = DC$. Point D is inside $\triangle ABC$, $\angle ABC = 40^\circ$, and $\angle ADC = 140^\circ$. What is the degree measure of $\angle BAD$?

3) Suppose m, n are positive integers such that $75m = n^3$. What is the minimum possible value of $m + n$?

4) The Dunbar family consists of a mother, a father, and some children. The average age of the members of the family is 20, the father is 48 years old, and the average age of the mother and children is 16. How many children are in the family?

5) An inner circle of radius 1 is surrounded by 4 outer circles of radius r, such that each outer circle is tangent to the inner circle and adjacent outer circles are tangent. What is r?

Hint: Ask for a picture if you are having a hard time visualizing the setup.

6) Two tour guides are leading six tourists. The guides decide to split up. Each tourist must choose one of the guides, but with the stipulation that each guide must take at least one tourist. How many different groupings of guides and tourists are possible?

7) A sphere is inscribed in a cube that has a surface area of 24 square meters. A second cube is then inscribed within the sphere. What is the surface area in square meters of the inner cube?

8) Suppose a number a satisfies the equation $4 = a + a^{-1}$. What is the value of $a^4 + a^{-4}$?

9) Right $\triangle ABC$ has $AB = 3, BC = 4, AC = 5$. Square $XYZW$ is inscribed in $\triangle ABC$ with X, Y on $\overline{AC}$, W on $\overline{AB}$, and Z on $\overline{BC}$. What is the side length of the square?

10) A set of 25 square blocks is arranged into a 5×5 square. How many different combinations of 3 blocks can be selected from the set so that no two are in the same row or column?

11) A triangle with side lengths in the ratio $3 : 4 : 5$ is inscribed in a circle of radius 3. What is the area of the triangle?

12) Integers a, b, c, d, not necessarily distinct, are chosen independently and at random from 0 to 2007, inclusive. What is the probability that $ad - bc$ is even?

Hint: What can you say about the parity of a, b, c, d?

13) A finite sequence of three-digit integers has the property that the tens and units digits of each term are, respectively, the hundreds and tens digits of the next term, etc., and the tens and units digits of the last term are, respectively, the hundreds and tens digits of the first term. For example, such a sequence might begin with terms $247, 475, 756, \cdots, 824$. Let S be the sum of all the terms in the sequence. What is the largest prime number that always divides S?

Hint: Look at smaller cases first, say sequences of length 3.

14) How many ordered pairs (m, n) of positive integers, with $m > n$, have the properties that their squares differ by 96?

Hint: If you know $m + n$ and $m - n$, do you know m, n?

15) A pyramid with a square base is cut by a plane that is parallel to its base and is 2 units from the base. The surface area of the smaller pyramid that is cut from the top is half the surface area of the original pyramid. What is the altitude of the original pyramid?

Hint: How does the altitude relate to the surface area?

16) Let n denote the smallest positive integer that is divisible by both 4 and 9, and whose base-10 representation consists of only 4's and 9's, with at least one of each. What are the last four digits of n?

Hint 1) Do you remember the divisibility rules for 4 and 9?

Hint 2) Without a calculator this is messy, but the following fact helps: the smallest answer works?

Review

Suppose we have an *alphabet* A (made up of *letters*). Today, A will be the binary alphabet (with letters $0, 1$) unless specified otherwise.

A *word* w (in the alphabet A), is simply a sequence of letters. For example, 001, 101, and 111 are all binary words, and 102 is a ternary word. Note, we'll allow words of length 0, denoted by ε.

A word v is a *subword* of a word w if v is contained in w. For example, $\varepsilon, 0, 1, 00, 01, 001$ are all subwords of 001.

We let $p_w(n)$ (called the *subword complexity* of w denote the number of subwords of w with length n. For example, if $w = 001$, we have

$$p_w(0) = 1,\ p_w(1) = 2,\ p_w(2) = 2,\ p_w(3) = 1.$$

If u, v are two words, then we use the natural notation uv to denote the word u followed by the word v. For example, if $u = 01$ and $v = 10$, then $uv = 0110$.

Let w be a word. Call u the *prefix* of w if u is all of w except for the last letter. Similarly, v is the *suffix* of w if v is all of w except for the first letter. For example, 0101 has prefix 010 and suffix 101; 1 has prefix and suffix equal to ε.

The graph G_n is constructed as follows. The vertices of the graph are all binary words of length $n-1$. There is an edge from vertex v to vertex w if the suffix of v is equal to the prefix of w. That is, there is a word u of length $n-1$ and letters x, y so that $v = xu$ and $w = uy$. Label this edge xuy.

Given a graph G, we construct the *line graph of* G (denoted $L(G)$) as follows: Every edge in G becomes a vertex in $L(G)$. If the end of edge a is the start of edge b in G, then there is an edge from (vertex) a to (vertex) b in L(G).

Sturmian Words

Call a word w a *Sturmian word* of order n if $p_w(m) = m+1$ for all $m \leq n$. Furthermore, it is called a *minimal Sturmian word* of order n if it is a Sturmian word and there is no shorter Sturmian word.

0) a) Show that the length of a Sturmian word of order n is at least $2n$.

b) (First attempt with $n = 3$) Start with G_3. Remove 4 edges (leaving 4 left) and call the remaining graph G'_3. Now construct a word using the method from Problem 4 on the previous page.

Show the following are all possible:

- The resulting G'_3 has no Eulerian path.
- The resulting G'_3 has an Eulerian path, leading to a word w with $p_w(3) = 4$, but $p_w(2) = 4$.
- The resulting G'_3 has an Eulerian path, leading to a word w which is a minimal Sturmian word of order 3.

c) Start with G_2. Remove one of the edges to get G'_2. Now compute $L(G'_2)$. If $L(G'_2)$ has 4 edges, set $G'_3 = L(G'_2)$. Else remove one edge from $L(G'_2)$ (ensuring we still have an Eulerian path) and call the remaining graph G'_3. Now construct a word w using the method from Problem 4 on the previous page. Do this a few times, and answer the following questions:

a) Is w a minimal Sturmian word?

b) What can you say about the subwords of w of length 2?

1) We can now construct minimal Sturmian words of order n as follows:

(1) Start with G_2 and create G'_2 by removing an edge from G_2.
(2) Now compute $L(G'_2)$. Removing an edge from $L(G'_2)$ if necessary (ensuring we still have an Eulerian path) create G'_3 a graph with 4 edges.
(3) Repeat the method in Step 2 to get $G'_4, \ldots, G'_n$.
(4) Now construct a word using the method from Problem 4 on the previous page.

a) Use the method above to construct minimal Sturmian words of order $4, 5$.

b) Argue that we actually are constructing minimal Sturmian words. That is, if we use the above method to construct w we have: i) the length of w is $2n$ and ii) $p_w(m) = m+1$ for all $m \leq n$.