INVARIANTS AND EXTREMES

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Today we're going to do some problems for their own sake. Whereas in previous meetings our problems were intended as a guide to some larger mathematical vista, today they are meant to show you problem-solving strategies that will help you along any path you travel.

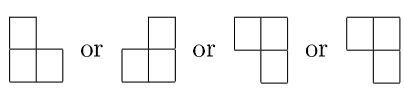
Invariance.

The problems below deal with processes that consist of the repetition of simples steps many many times. One of the best (and in fact only) ways to analyze and understand such processes is to find some feature of them that does not change as the steps are performed. Locating such **invariant** (that is, unchanging) features is more art than science, and is best learned by solving many problems.

1. Danny writes the numbers 1 through 2015 on a blackboard. He then chooses two of the numbers, say a and b, erases them both, and writes down $a-b$. He does this again and again until only one number is left. Can Danny choose his numbers carefully so that the number left at the end is 1?

2*. You are looking under a microscope and there are three kinds of bacteria: green, yellow, and blue. You are studying their unusual reproductive dynamics. You've noticed that sometimes a green bacterium divides producing one yellow and one blue specimen. Other times, three yellow and one blue specimen combine to produce two green ones. You leave the lab one night knowing that under the microscope there are 5 green, 7 yellow, and 3 blue bacteria. You arrive in the morning to find 8 green, 2 yellow, and 3 blue. Could this have happened naturally, or did someone tamper with your research?

3. You have an 8×8 chessboard and the numbers 1 through 64 are written in order in the squares. You are allowed the following moves: choose an L-shape consisting of three squares, like



and add 1 to each number in the L-shape. Is there a sequence of moves that will make all the numbers on the board be 99?

4*. The Fibonacci sequence starts with 1,1 and then each term is the sum of the two preceding terms. Thus the first few terms are 1, 1, 2, 3, 5, 8, 13, ...

What is the greatest common divisor (gcd) of two consecutive numbers on the Fibonacci sequence? (Hint: the gcd satisfies the following property: if $a > b$ then $\gcd(a, b) = \gcd(a - b, b)$. For instance, $\gcd(23, 7) = \gcd(23 - 7, 7) = \gcd(16, 7)$.)

5.** The numbers 1 through 10 are written in order on a blackboard. You are allowed two pick any two numbers and switch their positions. Can you put the numbers in reverse order using exactly 20 such switches?

Extremes.

When we are presented with a mathematical situation where we don't know many details — for instance, if we have a set of numbers but don't know what the numbers are — it is often useful to reason about *extreme* elements. These can be the smallest or largest number, the closest or furthest points on a map, the first or last time something happens, etc. The problems below can all be solved by considering the appropriate extreme object and reasoning about what properties it must satisfy.

7*. A car is going to drive on a circular road a thousand miles long. Along the road there are some gas stations, and the combined gas available at the stations is just enough for the car to go a thousand miles. Show that there is a gas station such that, if the car starts there with an empty tank, it can complete its journey without running out of gas.

8. A stranger on the bus makes you the following offer. She wants you to give her a set of positive integers with the following property: for every two numbers a, b in the set, there must be another number c in the set such that $a+b$ divides c. If you can make such a set using n numbers, she will pay you $1000/n$ dollars. How much money can you make?

9. Suppose you have a 100×100 chessboard and on each square is written a number. The arrangement has the following property: every number is exactly equal to the average of numbers surrounding it. Show that the only way this can happen is if all the numbers are the same.

10.** In a faraway country there are 100 cities. The distances between cities are all known and all different. Straight-line roads connect pairs of cities, in the following way. Two cities A and B are connected by a road if and only if A is the closest city to B or B is the closest city to A. Show that: (i) there is no city with more than five roads leaving it; and (ii) the roads don't form any polygons.

11*. You have a standard deck of cards and just completed your advanced Vegas card dealer training. You can do a perfect "riffle shuffle": you split the cards into two equal groups and then shuffle them together so that they exactly alternate. (This means that, if the cards in the first group are 1,2,3,... and those in the second group are A,B,C,... then after shuffling you will have one pile with the cards in the order 1,A,2,B,3,C,...)

If you keep doing the riffle shuffle over and over, are you guaranteed to eventually get back to the original ordering of the cards?