LAMC Intermediate I & II **February 8, 2015**

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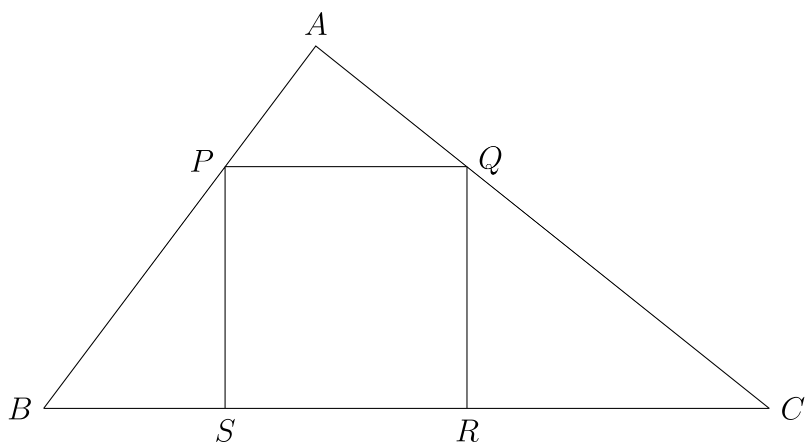
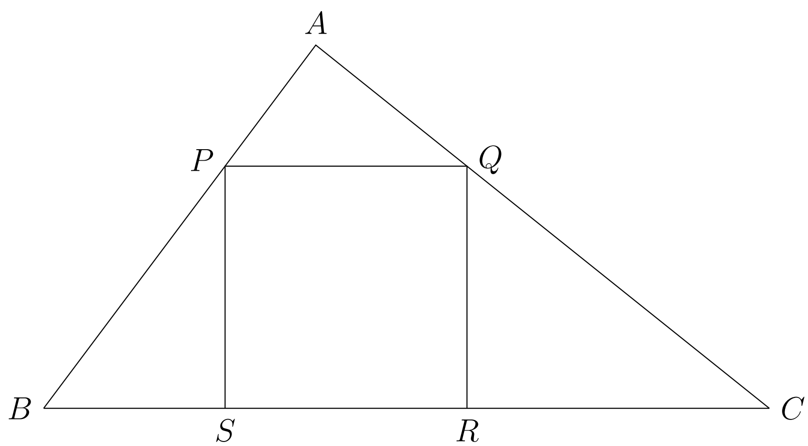
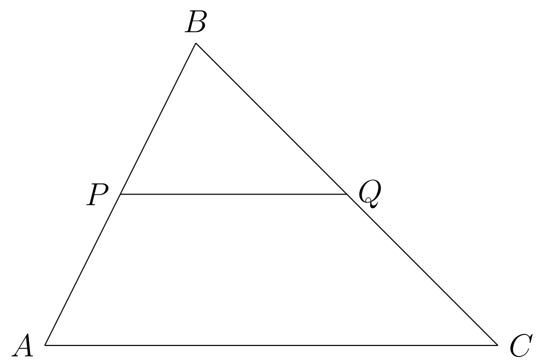
Problem 1 *The square $PQRS$ is inscribed in the triangle ABC as shown on the picture below. The length of the side BC is 12 cm. The side length of the square is 4 cm. Find the area of the triangle ABC.*



Problem 2 *The square $PQRS$ is inscribed in the triangle ABC as shown on the picture below. The length of the side BC is x cm. The side length of the square is y cm. Find the area of the triangle ABC.*

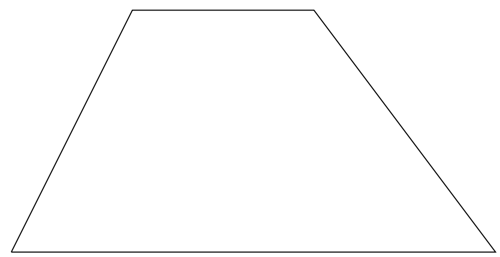


A *midline* of a triangle is a line joining centers of two sides.

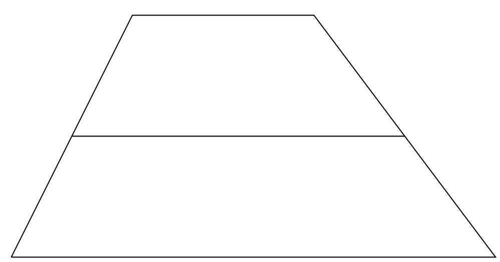


Problem 3 *Prove that the midline PQ is parallel to AC and that $|AC| = 2|PQ|$.*

A *trapezoid* is a quadrilateral having a pair of parallel opposite sides.

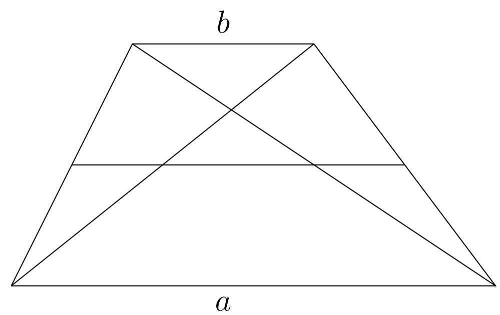


A midline of a trapezoid is the line connecting the centers of the non-parallel sides.



Problem 4 *The trapezoid on the picture above has the lower side of length a and the upper side of length b. What is the length of its midline?*

Problem 5 *Once again, the length of the lower side of the trapezoid is a and the length of its upper side is b. Find the length of the segment cut from the midline by the trapezoid's diagonals.*



Problem 6 *Prove that midpoints of the sides of any quadrilateral are vertices of a parallelogram.*

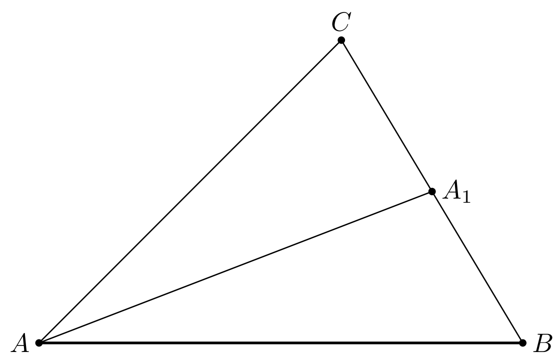
For what quads would the parallelogram be

- *a rectangle?*

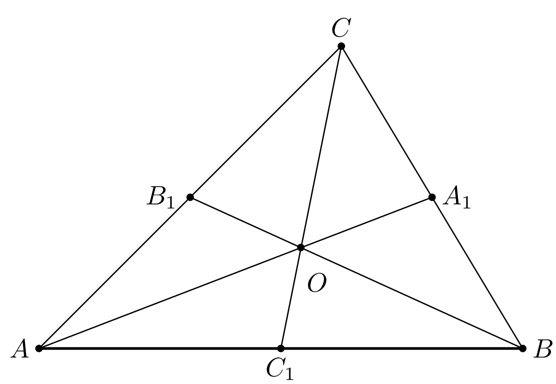
- *a rhombus?*

- *a square?*

A *median* of a triangle is a straight line segment joining a vertex of the triangle with the midpoint of the opposite side.

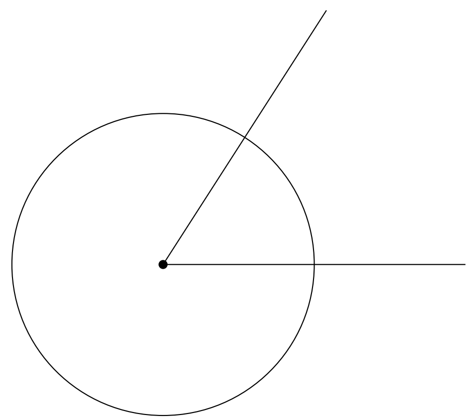


Problem 7 *Use similarity to prove that medians of a triangle intersect at one point that splits each of them in the ratio* $2:1$ *counting from the vertex.*

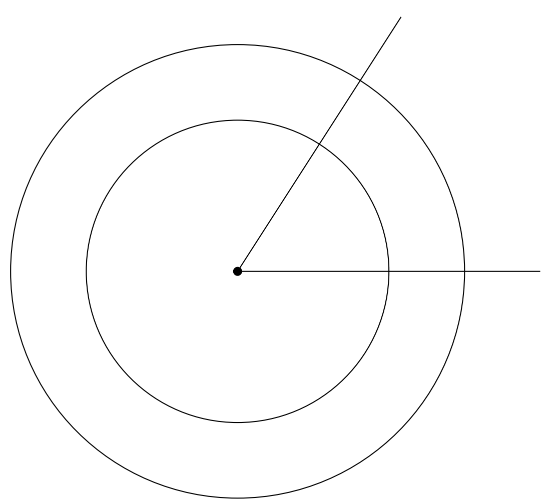


A natural unit of measuring angles, a radian

Consider an angle and a circle such that the vertex of the angle is the center of the circle. If the length of the arc the angle cuts off the circle is equal in length to the circle's radius, we say that the size of the angle is one *radian*.



The definition does not depend on the radius of the circle. Any two concentric circles are similar. Therefore, the ratio of the arc length to the radius is always the same.



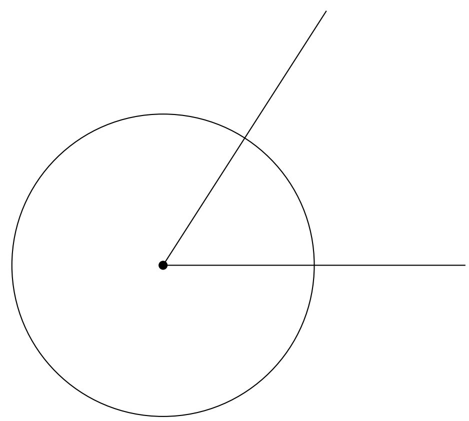
It is known that in the Euclidean plane, the length of a circumference of radius r is $2\pi r$, where $\pi = 3.1415926535897932...$ Therefore, there are 2π radians in the 360° angle.

Problem 8 *Convert the following angles to radians.*

270°	180°	120°	90°	60°	30°	15°

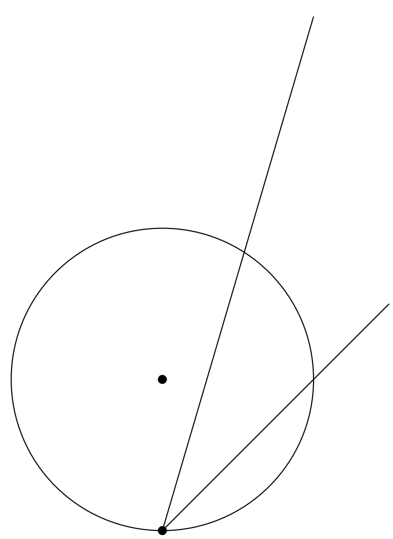
Problem 9 *Find the angles, in radians, of a right triangle with an angle $\pi/6$.*

An angle having its vertex in the center of a circle is called a *central* angle.



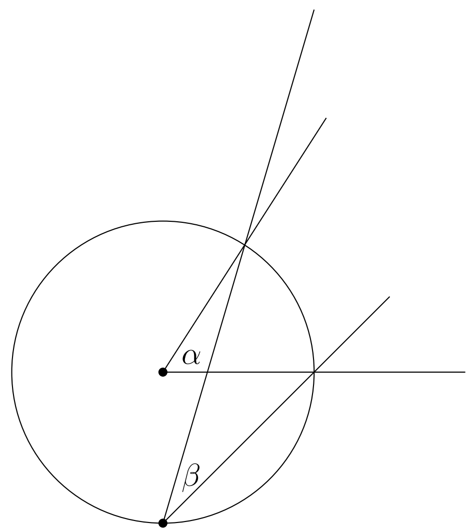
The size of a central angle, in radians, equals to the length of the arc it cuts off the circle, measured in the circle's radii.

An angle having its vertex on the circumference of a circle is called an *inscribed* angle.



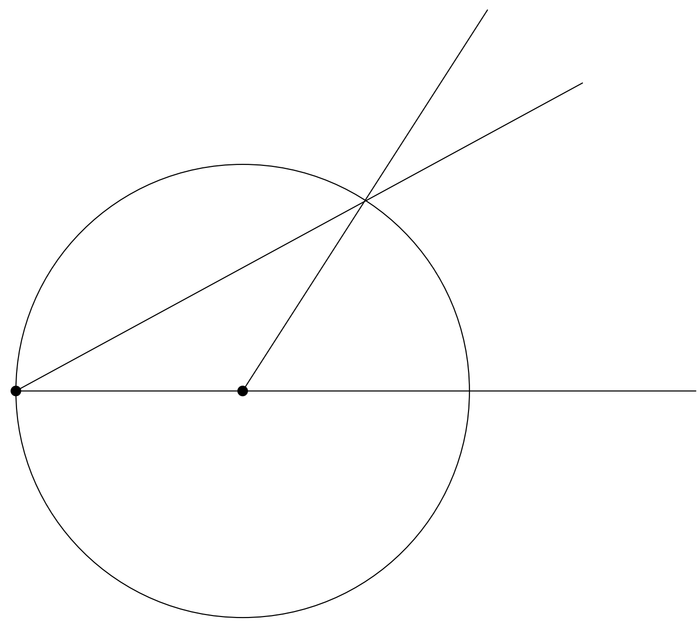
Theorem 1 *If an inscribed and a central angle cut the same arc off the circle, then the size of the central angle is twice the size of the inscribed one.*

$$\alpha = 2\beta$$

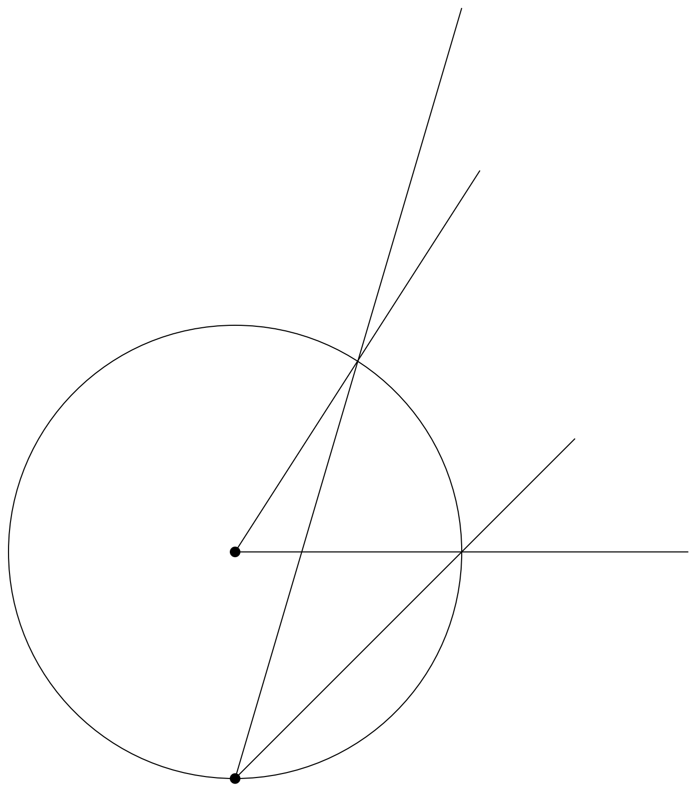


Problem 10 *Recall Thales' theorem. Draw the corresponding picture in the space below. Explain why the theorem of Thales is a particular case of Theorem 1.*

Problem 11 *Prove Theorem 1 in the special case when a side of the inscribed angle passes through the center of the circle. If you need to measure angles, use radians, not degrees.*



Problem 12 *Prove the general case of Theorem 1.*



If you are finished doing all the above, but there still remains some time ...

Problem 13 *While playing in the park, Ivan and Peter came to a large round clearing surrounded by a ring of aspen trees. The boys decided to count the trees. As they walked around the clearing counting, Ivan's 20th tree turned out to be Peter's 7th, while Ivan's 7th was Peter's 94th. How many trees were growing around the clearing?*

Problem 14 *The length of the Earth's equator is 40,075 km. A rope is tied tight around the Equator. Then they add a 1-meter-long piece to the rope and stretch it evenly above the Earth's surface. Assuming that the Earth is a perfect ball, would a cat be able to squeeze itself between the Earth and the rope?*

Problem 15 *A train moves in one direction for 5.5 hours covering any 100-mile part of the journey in 1 hour. Is the train necessarily moving at a constant speed? Is the train's average speed for the entire journey necessarily equal to 100 mph?*