Let O pe o point in the plane, and $r > 0$ a fixed number. Call $\mathcal{C}$ be the circle with center O and radius r.

For each point $A \neq O$ in the plane, define the inverse of A, denoted by A', the unique point on the half line OA such that $OA \cdot OA' = r^2$.

Exercise 1) What is the image of the circle $\mathcal{C}$ through inversion?

Exercise 2) What is the image of a line passing through O, through inversion?

Exercise 3) The inverse of a line not passing through O is a circle passing through O.

Exercise 4) The inverse of a circle passing through O is a line not passing through O. (DEDUCE THIS IMEDIATELY FROM 3, without any other computations!)

Exercise 5) Suppose we have two circles, tangent at O. From Exercise 4, we know that the inverses of the two circles are two lines. Prove that these two lines are paralel.

Exercise 6) The inverse of a circle not passing through O is a circle not passing through O.

Exercise 7) Where does the inverse of a point outside circle $\mathcal{C}$ lie?

Exercise 8) Let O be a point, and $mathcalD$ a circle that does not contain O. Prove that there exists and $r > 0$, such that inversion of center O and $r > 0$ keeps circle $\mathcal{D}$ invariant.

Exercise 9) (Appolonius' circles - particular case) Suppose we have three circles, C_1, C_2, C_3, disjoint and none contains the others. Suppose C_1 and C_2 are tangent at a point O. Construct a circle D that is tangent to all three of them.

Exercise 10) Suppose we have 4 circles, C_1, C_2, C_3, C_4, such that C_1 is tangent to C_2, C_2 to C_3, $C_3 to C_4$ and C_4 to C_1. Prove that four the tangency points form a cyclic quadrilateral.

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Exercise 11) (AMC 12B, 2013, Prb 22): Let $m, n > 1$ be integers. Suppose that the product of the solutions for x of the equation

$$8(log_n x)(log_m x) - 7log_n x - 6log_m x - 2013 = 0$$

is the smalles possible integer. What is $m + n$.