

# Math Circle, HighSchool 2

January 11, 2015

1. In a triangle with integer side lengths, one side is three times as long as a second side, and the length of the third side is 15. What is the greatest possible perimeter of the triangle?
2. From among six couples, a committee of 5 members is to be formed. If the selected committee has no couple, then in how many ways can the committee be formed?
3. (AMC 2009 12B, prb. 23) A region  $S$  in the complex plane is defined by

$$S = \{x + yi \mid -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1\}$$

A complex number  $z$  is chosen uniformly at random from  $S$ . What is the probability that  $(\frac{3}{4} + \frac{3}{4}i)z$  is also in  $S$ ?

4. Heron's formula gives a way of computing the area of a triangle, knowing all its sides. It says the following: If a triangle  $ABC$  has sides of length  $a, b, c$ , denote by  $s$  the semiperimeter, that is,  $s = \frac{a+b+c}{2}$ . The area of the triangle is  $A = \sqrt{s(s-a)(s-b)(s-c)}$ . Now solve the following problem, given at AMC 2002 12A, Problem 23: In triangle  $ABC$ , side  $AC$  and the perpendicular bisector of  $BC$  meet in point  $D$ , and  $BD$  bisects angle  $ABC$ . If  $AD=9$  and  $DC=7$ , what is the area of triangle  $ABD$ ?
5. (AMC 12A, 2011, prb 18) Suppose that  $|x + y| + |x - y| = 2$ . What is the maximum possible value of  $x^2 - 6x + y^2$ ?
6. Find positive integers  $a, b$  such that  $a^4 + 4b^4$  is prime. (HINT:  $a^2 + 2ab + 2b^2$  is a factor of  $a^4 + 4b^4$ . Having this information, how can you find the other factor?)
7. (Pick's theorem): Suppose we have a polygon with vertices sitting on a grid. Try to find a formula for the area of the polygon, in terms of the number  $I$  of grid points inside the polygon, and  $B$ , the number of grid points sitting on the boundary (edges) of the polygon.
8. Triangle  $ABC$  has  $AB = 13$ ,  $BC = 14$ , and  $AC = 15$ . The points  $D$ ,  $E$ , and  $F$  are the midpoints of  $AB$ ,  $BC$ , and  $AC$  respectively. Let  $X \neq E$  be the intersection of the circumcircles of  $BDE$  and  $CEF$ . What is  $XA + XB + XC$ ?