Trigonometry 2

Math Circle Advanced Group

12/7/14

Remember, if you have a right triangle with legs a, b and hypotenuse c, and the acute angle next to a is θ, then $\sin\theta = \frac{b}{c}$, and $\cos\theta = \frac{a}{c}$.

Problem 1 Draw the picture associated to these definitions.

In our problems today, we will label the three sides of a triangle as a, b, c; the angle opposite a will be A, the angle opposite b will be B, and the angle opposite c will be C. The altitude to side a from angle A will be h_a, the altitude to side b from angle B will be h_b, and the altitude to side c from angle C will be h_c.

Problem 2 Draw two triangles with all of their altitudes, sides, and angles labeled as above. One triangle should be acute, the other should be obtuse.

Problem 3

Assume our triangle is acute.

Find two ratios in terms of a, b, c, h_a, h_b, h_c that are equal to $\sin A$

Find two ratios in terms of a, b, c, h_a, h_b, h_c that are equal to $\sin B$

Find two ratios in terms of a, b, c, h_a, h_b, h_c that are equal to $\sin C$

Problem 4

Remember, the law of sines says that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Use the first two parts of problem 3 to show that, for an acute triangle, $\frac{a}{\sin A} = \frac{b}{\sin B}$

Use the second two parts of problem 3 to show that, for an acute triangle, $\frac{b}{\sin B} = \frac{c}{\sin C}$

Problem 5

Repeat problems 3 and 4 for obtuse triangles. Remember that the sine of an obtuse angle is equal to the sine of its supplementary angle ($\sin(A) = \sin(180^\circ - A)$)

Problem 6

Use the law of sines to prove that larger sides are opposite larger angles. Be careful; this is a little tricky with obtuse triangles.

Problem 7

Verify the law of sines for $45^\circ - 45^\circ - 90^\circ$ and $30^\circ - 60^\circ - 90^\circ$

Problem 8

While geometry tells us that two sides and an included angle, or two angles and a side, are sufficient to uniquely determine a triangle, it doesn't tell us how to figure out the other sides and angles of that triangle. The law of sines, on the other hand, does. Use the law of sines to approximate the remaining sides and angles for the following triangles (using calculators when necessary)

(i) $A = 50^\circ, B = 70^\circ, c = 10$

(ii) $A = 55^\circ, B = 65^\circ, a = 12$

(iii) $A = 30^\circ, B = 45^\circ, a = 6$

(iv) $A = 50^\circ, a = 8, b = 10$

Problem 9

Remember that Side-Side-Angle isn't sufficient to prove congruence of triangles. Is the triangle from 8(iv) uniquely determined? What are the possible angle measurements that fit the given data?

Problem 10 Extended Law of Sines

Suppose our triangle is inscribed in a circle of radius R (that is, the vertices of the triangle all lie on the circle). Show that the extended law of sines holds:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

Problem 11

Find a formula for the area of a triangle in terms of:

(i) a, b, and C

(ii) a, B, and c

(iii) A, b, and c

Problem 12

fix the side lengths a and b of a triangle; what is the shape of such a triangle with largest possible area?

Problem 13

The law of cosines is a generalization of the Pythagorean theorem. It says that if $C \leq 90°$, $c^2 = a^2 + b^2 - 2ab\cos C$

Prove the law of cosines.

Come up with an alternate version of the law of cosines, that holds when $C > 90°$.

Problem 14

If $s = (a+b+c)/2$, use previous problems to prove that the area of a triangle is $\sqrt{s(s-a)(s-b)(s-c)}$