Geometry Kaleidoscope

October 24, 2014

Problem set 1: squares and circles

1. Given a square $ABCD$, find the set of all points on the plane such that the sum of the distances from each point to AB and to CD equals to the sum of the distances from the same point to BC and AD.

2. A coin rolls around another coin and makes a full circle. How many times does the coin rotate? First, make a guess. Then, make an experiment. Finally, find an explanation.

3. A ladder standing next to a wall starts sliding down and eventually falls. What is the trajectory of a kitten sitting in the middle of the ladder? (Assume the kitten is very calm and does not jump when the ladder starts moving).

4. A circle of radius R rolls inside of a circle of radius $2R$. What is the trajectory of a point on the smaller circle?

5. Given a triangle, inscibed a square into it in the following sense: draw a square inside the triangle such that two of its vertices are on the base of the triangle and two vertices are on the other two sides of the triangle.

Problem set 2: Lighthouse and other problems

1. How far can you see from a lighthouse of height h assuming the Earth's radius is R? Given that $R \approx 6,400km$ and $h \approx 16m$ for the Kileau lighthouse (located on the North shore of the island of Kauai), estimate the maximal distance at which you can see a ship from the top of this lighthouse.
2. What is the biggest number of squares on a 6×6 checkerboard that you can color in such a way that none of them are touching each other (even at a vertex)?
3. Cut a rectangle into two pieces which can be put together to make a triangle.
4. Cut a triangle into four pieces which can be put together to make a rectangle.

Problem set 3: examples and computations

1. What is the biggest number of sides that a polygon obtained as an intersection of a triangle and a quadrilateral can have?

2. Give an example of a triangle cut into

   (a) three congruent triangles;

   (b) four congruent triangles;

   (c) five congruent triangles;

3. A sail has the shape of a quadrilateral $ABCD$ with angles $\angle A = \angle B = \angle D = 45^\circ$. Given that $|AC| = 4$ meter, find the area of the sail.

4. Suppose that a square of side length a is inscribed into a circle. Take one of the four segments between the circle and the square and inscribe a new square into this segment. What is the side length of this square?

Problem set 4 (Polygons)

1. What types of polygons can you get when you intersect a cube by a plane? Consider all possible cases.
2. What is the sum of external angles of an n-gon (polygon with n sides)? How does it change with the number of sides? Find a good explanation of your result.
3. What kind of quadrilateral do you get if you connect the four midpoints of the sides of an arbitrary quadrilateral?
4. What is the biggest number of parts you can get when drawing n straight lines on the plane? What is the biggest number of parts you can get for $n = 2014$?

Problem set 5: Circles

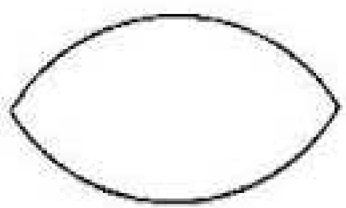
1. Suppose you are walking around the Earth's equator. What is the difference between the distance travelled by your head and your feet? (First, make a guess. The, make a computation. You can use the height of 1.6m (which is nearly $5'3''$) in the computation. Your answer should be a number.

2. You tie a belt around the middle of a (perfectly round) orange. Then you increase the length of the belt by one meter and pull it away from the orange so that it is at the same distance from the orange on all sides. Now you do the same thing with the (perfectly round) Earth: tie a belt around the equator; increase its length by one meter, and pull it away from the surface to be at the same distance in all directions. Will the belt be closer to the Earth or to the orange? For example, would a mouse be able slip under the belt tied around the Earth? (First make a guess, then make a computation).

3. If you cool down steel by $1°C$, it shrinks by a factor $\frac{1}{100{,}000}$ (i.e., loses this fraction of its length). A steel wire is tied around the Earth's equator. If you cool it by $1°C$, the wire will shrink and sink into the Earth to some depth. How far will it sink? Make a guess, then make a computation.

Problem set 6: Tying Goats

1. (Warm-up) What section of a pasture is consumed by a goat if the goat is tied to a single stake planted in the pasture?
2. You take a walk on a field hodling a goat on a 1-meter-long rope. You are walking along the path which is a rectangle with dimensions 3 meters $\times$ 5 meters. What is the section of the field that goat has accesses to while on the walk?
3. How can you constraint a goat to an eye shaped field:

   In other words, how can the goat be tied using ropes and stakes so that it can eat only the grass in the eye-shaped field above?
4. A rope has been stretched between two stakes in a field. A goat is tied to this rope with another rope that is free to slide aong the first rope. What is the shape of the portion of the field the goat can eat?
5. Suppose that you know how to constraint the goat to a shape F and how to contstraint the goat to a shape G. Explain how to constraint the goat to the intersection of F and G.
6. Use the previous problems to constraint the goat to a field in the shape of a semicircle.
7. Use the previous problems to constraint the goat to a filed in the shape of a s square.

Problem set 7: Bridges and Roads

1. As you are standing near a river (but not at the river bank!), you see a fire at a house on the same side of the river. You want to put out the fire as fast as possible, so you run to the river to get water and then to the house. At what point on the river should you get the water so that the distance you cover before you get to the burning house is minimized?

2. You are standing on a path connecting you with a tall tree. You want to place a mirror at some point on the path in such a way that when you look into the mirror you see the top of the tree. Where should you place the mirror?

3. Two cities are located on opposite sides of a river with parallel straight sides. You need to build a road between the cities. The road includes a bridge that needs to be perpendicular to the river. Where should the bridge be located so that the total length of the road is the shortest possible?

4. To get to school from your house, you need to cross two streets which intersect perpendicularly. You can walk to a street in any way you like, but have to cross a street perpendicularly to the street. What is the shortest path from your house to school? Make a picture and consider all possible cases.

5. Two towns are separated by two rivers. Each river has parallel banks. However, the two rivers are not parallel to each other. Where should bridges that cross the rivers perpendicularly to the banks be built so that the distance between the two towns is as short as possible.